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A large class of strategy-proof exchange rules with single-peaked preferences

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Abstract

We study the classical house exchange problem (Shapley and Scarf (1974)) and identified a large class of rules, each of which is strategy-proof, efficient, and individually rational with single-peaked preferences. These rules are generalizations of Gale's top trading cycles rule: In each step a subset of houses are allowed to be traded along top trading cycles and in particular, the next step subset of trading houses may depend on the exchanges happened already. We believe that the flexibility introduced by this class of rules is desirable when the designer faces some context-specific requirements.

Keywords: Top trading cycles; strategy-proof; single-peaked;

JEL Classification: C78, D71.

1 Introduction

We study the so-called "house exchange problem" where some privately owned objects need to be reallocated to their owners without money transfer. For this problem Shapley and Scarf (1974) introduced the top trading cycles rule (TTC hereafter) and attributed it to David Gale. The TTC rule always selects a core allocation, which implies efficiency and individual rationality. Later it was proved to be strategy-proof by Roth (1982). After that, it was shown by Ma (1994) to be the unique strategy-proof, efficient, and individually rational rule.[1]

Since then the TTC rule has been in the center of designing mechanisms for allocating indivisible objects without transfer and has been adapted to deal with variants of the problems. Examples include the hierarchical exchange rules (Pápai (2000)) and the trading cycles rules (Pycia and Ünver (2017)) for the house allocation problems and the so-called you-request-my-house I-get-your-turn rules (Abdulkadiroğlu and Sönmez (1999)) for the house allocation problems with existing tenants. It has also been studied and compared to some other rules for the school choice problems (Abdulkadiroglu and Sönmez (2003)) and the kidney exchange

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[1]Some other proofs of the uniqueness can be found in Svensson (1999), Anno (2015), and Sethuraman (2016).

problems (Roth et al. (2004)). Recently it has been adapted for the problem of trading fractional shares and probabilities rather than intact objects. Examples include Kesten (2009), Aziz (2015), and Altuntas and Phan (2017) among others.

In this paper, we revisit the house exchange problem and notice that some flexibility required by problem-specific context can not be provided by the TTC rule. Below is an example, which indicates that there are situations where we want to adjust the endowment before applying the TTC rule.

Example 1. Let $I = \{1, 2, 3, 4\}$ be four agents and each agent i owns a house h_i. Assume that their preferences are P_i's as follows.

$$\begin{array}{rcccc} P_1: & h_4 & h_3 & h_2 & h_1 \\ P_2, P_3, P_4: & h_1 & h_2 & h_3 & h_4 \end{array}$$

where P_1 reads "h_4 is strictly better than h_3, h_3 is strictly better than h_2, and h_2 is strictly better than h_1." If we employ the TTC rule, agent 1 and 4 exchange their houses and agent 2 and 3 remain unchanged, as below

$$TTC(P) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ h_4 & h_2 & h_3 & h_1 \end{pmatrix}$$

However, suppose agent 2 is a senior citizen and we want the final allocation to favor him, which in the current case means that we want him to get h_1, the only way to make him better off. Notice that agent 1 treats h_2 as acceptable. Then the question becomes "Should we let agent 1 and 2 exchange their endowments before the application of the TTC rule?" If we do so, the allocation will be as follows, which achieves our goal without sacrifice in efficiency or individual rationality.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ h_4 & h_1 & h_3 & h_2 \end{pmatrix}$$

However, with slight reflexion, we notice a problem: how about incentive compatibility? In particular, we have to announce the rule before agents reporting their preference. If we announce the rule as that, in the first step, we give agents 1 and 2 a chance to exchange their endowments and in the second step the TTC rule is implemented, is there any incentive for any agent to misrepresent her preference?

Motivated by the above observation, we introduce a large class of rules, called dynamic TTC rules, within which Gale's TTC rule is a special case. A dynamic TTC rule selects the final allocation by several steps. In each step the TTC rule restricted to a subset of houses, equivalently a subset of agents, is implemented. As in the above example, in the first step, the TTC rule restricted to h_1 and h_2, equivalently agents 1 and 2 as they own these two houses, is implemented and in the second step the TTC rule restricted to H is implemented. Generally, the next step subset of trading houses may depend on the temporary allocation from the previous steps. Such path-dependence is illustrated below.

Example 1 (Continued)**.** In the first step, the TTC rule restricted to h_1 and h_2 will be implemented. After that, if agents 1 and 2 exchange their houses, the TTC rule restricted to

$\{h_1, h_2, h_3\}$ will be implemented. If otherwise, the TTC rule restricted to H will be implemented. For the preferences we listed, the allocation selected will be

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ h_3 & h_1 & h_2 & h_4 \end{pmatrix}$$

Another class of situations where such a dynamic TTC rule is called upon is that some agents are delayed who can not show up until tomorrow while the others who are ready for the exchange may request trading among themselves today. The example below illustrates such a situation.

Example 2. Let $I = \{1, 2, 3, 4\}$ be four agents and each agent i owns a house h_i. Let in addition their preferences be as follows

$$\begin{array}{rcccc} P_1: & h_4 & h_3 & h_2 & h_1 \\ P_3: & h_2 & h_1 & h_3 & h_4 \\ P_2, P_4: & h_1 & h_2 & h_3 & h_4 \end{array}$$

Suppose due to some personal issue agent 4 can not show up until tomorrow while all the others are already ready for exchange. Now if we announce that TTC rule restricted to $\{h_1, h_2, h_3\}$ will be implemented today and when agent 4 comes tomorrow, Gale's TTC rule will be implemented, the final allocation will be

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ h_4 & h_1 & h_2 & h_3 \end{pmatrix}.$$

If instead we insist that we wait for agent 4 and implement Gale's TTC rule tomorrow, the allocation will be

$$TTC(P) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ h_4 & h_2 & h_3 & h_1 \end{pmatrix}.$$

It is evident that the two-step dynamic TTC rule selects an allocation which every one of the present agents treats at least as good as the allocation selected by Gale's TTC, with someone strictly bettered off. This method seems to be justified especially when we consider the cost incurred by letting them wait for the whole day.

However, the same problem arises: is this dynamic TTC rule strategy-proof?

Since in every step a TTC rule restricted to a subset of trading houses is implemented, whenever an agent exchanges her holding house with some other, she gets better off and hence every such dynamic TTC rule always selects an individually rational allocation.

In order to secure efficiency, we require that the procedure of a dynamic TTC always terminates with a TTC restricted to the whole set H. This is a natural specification in relevant applications. A counterexample is the second dynamic TTC rule in Example 1, in which a path ends with $\{h_1, h_2, h_3\}$, the allocation selected is inefficient. However, notice that after several steps of TTC restricted to various subsets of trading houses, we should expect that many agents will not exchange with anybody in the last step although they are allowed to do so. As in the dynamic TTC rule in Example 2, although in the next day every body is allowed to trade, only agents 1 and 4 will exchange their houses.

Knowing that every dynamic TTC rule is efficient and individually rational, we focus on strategy-proofness. To do so, notice a unique feature of dynamic TTC rules that is not in any existing generalization of TTC: incentive compatibility now involves essentially trade-offs between temporary improvement in a certain step and the opportunities of exchanges in future steps. In particular, it might be profitable for an agent to sacrifice the available improvement in a certain step in order to make herself stand on a preferred trading condition for some future exchanges. Such a unique feature arises because generally in a certain step not all the houses are available for exchange, which is not true in any of the generalizations mentioned in the opening of the paper. In those generalizations, all the houses are available for exchanges from the beginning.

Due to the fact that every dynamic TTC rule is efficient and individually rational, an implication of Ma (1994)'s characterization is that, on the universal domain, the class of dynamic TTC rules collapses to Gale's TTC rule when strategy-proofness is imposed.

However, if the admissible preferences are single-peaked, an interesting subclass of dynamic TTC rules emerges when we impose strategy-proofness. In a house exchange problem, single-peakedness may be defined with respect to the size of houses or the market value of them. In this paper, we assume that houses are ordered according to size. Then single-peakedness means that an agent's satisfaction from a house increases with size but just to a certain value and then decreases. Given that the houses are ordered by market value, single-peakedness means that, according to her financial status, an agent calculates her optimal expenditure on housing. Then a house is better if its market value is closer to her optimal expenditure.

It is evident that not every dynamic TTC rule is strategy-proof on the single-peaked domain. However, we found that as long as every subset of trading houses is a neighborhood, the corresponding dynamic TTC rule is strategy-proof, where a neighborhood refers to a subset of houses identified by an interval of size. Bade (2017) recently introduced a rule on the single-peaked domain called "crawler", which is shown to be a special case of our dynamic TTC rules with neighborhoods.

The next interesting question is naturally: are neighborhood trading subsets necessary for a dynamic TTC rule to be strategy-proof? We answer this question with the restriction to those rules without path-dependence. In other words, we focus on the dynamic TTC rules where the set of trading houses in the next step does not depend on the temporary allocation from the previous steps. Within these rules, we show the necessity of neighborhoods up to outcome-equivalence. In particular, whenever a dynamic TTC rule without path-dependence is strategy-proof it is equivalent to a dynamic TTC rule with neighborhoods, in the sense that, for each admissible economy, they select the same allocation.

Finally we address the question: can we expand the single-peaked preference domain while reserving strategy-proofness of the dynamic TTC rules with neighborhoods? The answer is definitely in negative. In particular, we show whenever a non-single-peaked preference is made admissible, there is some dynamic TTC rule with neighborhoods which becomes manipulable.

The sequel of the paper is organized as follows. Section 2 introduces the formal definitions and notations. Section 3 defines the class of dynamic TCC rules. Section 4 studies the strategy-proofness of these rules and Section 5 concludes.

2 House Exchange Problems

Let $I \equiv \{1, \cdots, n\}$ be the set of agents and $H \equiv \{h_1, \cdots, h_n\}$ the set of houses. **An allocation** is an one-to-one mapping $m : I \to H$ and the set of all allocations is denoted as M. The **endowment** is an allocation, denoted as $e \in M$. Without loss of generality we fix it as such that $e(i) = h_i$ for all $i \in I$.

Each agent $i \in I$ is equipped with a strict preference P_i on houses, i.e., an antisymmetric, transitive, and complete binary relation on H.[2] For an arbitrary preference P_i and an arbitrary nonempty subset $\hat{H} \in 2^H \backslash \emptyset$, let $\tau(P_i, \hat{H})$ denote the favorite object in $\hat{H}$, i.e., $\tau(P_i, \hat{H}) = h$ such that $h \ P_i \ h'$ for all $h' \in \hat{H} \backslash \{h\}$. In particular, let $\tau(P_i) \equiv \tau(P_i, H)$.

Let $\mathcal{P}$ be the set of all strict preferences and we call $\mathcal{P}$ the universal preference domain. For an specific class of allocation problems, the admissible preferences might not be the entire universal preference domain but a subset $\mathcal{D} \subset \mathcal{P}$. We call this subset the **domain** of admissible preferences.

An admissible economy is a tuple (I, H, P, e), where $P = (P_i)_{i \in I} \in \mathcal{D}^I$ is a profile of admissible preferences, one for each agent. Throughout the paper we fix I, H, and e. Hence we denote an admissible economy simply as $P \in \mathcal{D}^I$. An exchange rule, or simply **a rule**, is a mapping $\varphi : \mathcal{D}^I \to M$ that selects for each admissible economy an allocation.

We will impose some axioms on a desirable rule. The first deals with incentive compatibility, strategy-proofness, which requires that reporting the true preference in the direct revelation game is always a weakly dominant strategy.

Definition 1. *A rule $\varphi : \mathcal{D}^I \to M$ is **strategy-proof** if and only if, for all $P \in \mathcal{D}^I$ and all $P'_i \in \mathcal{D}$, $\varphi_i(P) \neq \varphi_i(P'_i, P_{-i})$ implies $\varphi_i(P) \ P_i \ \varphi_i(P'_i, P_{-i})$.*

The second is efficiency in Pareto sense: the allocation can not be improve in a feasible way that some agent gets a better house without hurting any other agent.

Definition 2. *For an arbitrary admissible economy, $P \in \mathcal{D}^I$, an allocation $x \in M$ is **efficient** if and only if $\nexists y \in M$ such that $y \neq x$ and $\forall i \in I$, $y(i) \neq x(i) \Rightarrow y(i) \ P_i \ x(i)$. A rule is efficient if it selects for each admissible economy an efficient allocation.*

The third requires that no agent gets a house worse than her endowment, which encourages participation.

Definition 3. *For an arbitrary admissible economy, $P \in \mathcal{D}^I$, an allocation $x \in M$ is **individually rational** if and only if $\forall i \in I$, $x(i) \neq h_i \Rightarrow x(i) \ P_i \ h_i$. A rule is individually rational if it selects for each admissible economy an individually rational allocation.*

3 Dynamic Top Trading Cycles Rules

As a preparation for the definition of our rules, we adapt Gale's TTC rule by restricting the set of houses that are allowed to be traded, which in fact also restricts the set of agents who are allowed to trade.

[2] A binary relation P_i is antisymmetric if $h \ P_i \ h'$ and $h' \ P_i \ h$ imply $h = h'$, transitive if $h \ P_i \ h'$ and $h' \ P_i \ h''$ imply $h \ P_i \ h''$, and complete if either $h \ P_i \ h'$ or $h' \ P_i \ h$ holds for arbitrary h and h'.

Definition 4. *Let* $TTC : \mathcal{D}^I \times M \times (2^H \backslash \emptyset) \to M$ *be such that for an arbitrary preference profile* $P \in \mathcal{D}^n$*, an arbitrary allocation* $m \in M$*, and a nonempty subset of houses* $\hat{H} \subset H$*, let* $\hat{I} \equiv \{i \in I : m(i) \in \hat{H}\}$ *and* $\hat{m} = TTC(P, m, \hat{H})$ *is a new allocation such that* $\hat{m}(i) = m(i)$ *for all* $i \in I \backslash \hat{I}$ *and* $\hat{m}(i)$ *for* $i \in \hat{I}$ *is specified below*

- *Let* $\tilde{I}_1 = \hat{I}$ *and* $\tilde{H}_1 = \hat{H}$.
- *Round 1: Each agent* $i \in \tilde{I}_1$ *points to the owner of her favorite house in* $\tilde{H}_1$*, i.e.,* $i \in \tilde{I}_1$ *points to* $m^{-1}(\tau(P_i, \tilde{H}_1))$*. Since* $\tilde{I}_1$ *is finite, there is at least one cycle. Let* $C_1 \subset \tilde{I}_1$ *denote the collection of agents who are involved in a cycle. Let for each* $i \in C_1$, $\hat{m}(i) = \tau(P_i, \tilde{H}_1)$*. In addition, let* $\tilde{I}_2 = \tilde{I}_1 \backslash C_1$ *and* $\tilde{H}_2 = \tilde{H}_1 \backslash \{m(i) : i \in C_1\}$*. If* $\tilde{I}_2 \neq \emptyset$*, proceed to the next round.*
- $\vdots$
- *Round t: Each agent* $i \in \tilde{I}_t$ *points to the owner of her favorite house in* $\tilde{H}_t$*, i.e.,* $i \in \tilde{I}_t$ *points to* $m^{-1}(\tau(P_i, \tilde{H}_t))$*. Since* $\tilde{I}_t$ *is finite, there is at least one cycle. Let* $C_t \subset \tilde{I}_t$ *denote the collection of agents who are involved in a cycle. Let for each* $i \in C_t$, $\hat{m}(i) = \tau(P_i, \tilde{H}_t)$*. In addition, let* $\tilde{I}_{t+1} = \tilde{I}_t \backslash C_t$ *and* $\tilde{H}_{t+1} = \tilde{H}_t \backslash \{m(i) : i \in C_t\}$*. If* $\tilde{I}_{t+1} \neq \emptyset$*, proceed to the next round.*

Since in each round there is at least one cycle formed and $\hat{I}$ *is finite, this algorithm terminates in finite rounds.*

We define a trading tree which is later used as a parameter to define a dynamic TTC rule.

Let (V, Q) be a rooted tree,[3] where $V \neq \emptyset$ is the vertex set and $Q \subset V \times V$ is the set of arcs. Let v^0 denote the root of the tree, i.e., $\nexists v \in V$ such that $(v, v^0) \in Q$. For each $v \in V \backslash \{v^0\}$ different from the root, there is a unique path from the root to v. We denote this path as $P(v^0, v)$, which is a sequence of a subset of the vertex set $v_1, \cdots, v_k, v_{k+1}, \cdots, v_K$ such that $v_1 = v^0$, $v_K = v$, and $(v_k, v_{k+1}) \in Q$ for all $k = 1, \cdots, K-1$. We call a vertex v terminal if there is no vertex v' such that $(v, v') \in Q$. In addition let Z be the set of terminal vertices. For each nonterminal vertex $v \in V \backslash Z$, let $Q_v = \{(v, v') \in Q\}$ be the set of arcs directed away from v.

Let the rooted tree be labeled. In particular, the vertex set V is labeled by nonempty subsets of houses, i.e., $\mathcal{V} : V \to 2^H \backslash \emptyset$ and the arc set is labeled by allocations, i.e., $\mathcal{Q} : Q \to M$.

We call a labeled rooted tree $T = (V, Q, \mathcal{V}, \mathcal{Q})$ **a trading tree** if the following two conditions are satisfied. Condition C1 concerns the labeling of vertices and requires that a vertex is labeled by the whole set H if and only if it is terminal.[4] Conditions C2 concerns the arcs. It can be seen as a composition of two restrictions. First, the number of arcs originated from each non-terminal vertex is exactly $n!$. Second, the labels of these $n!$ arcs are distinct from each other. Formally:

C1: For each vertex $v \in V$, $\mathcal{V}(v) = H$ if and only if $v \in Z$.

C2: For each non-terminal vertex $v \in V \backslash Z$, $\mathcal{Q}$ restricted to Q_v is an one-to-one mapping from Q_v to M.

[3] A rooted tree is a tree with a vertex designated as the root so that a distance from the root can be calculated for each node and an arc can be treated as directed away from the root.

[4] As we discussed in the introduction, we impose this condition in order to guarantee efficiency.

Fix a trading tree $T = (V, Q, \mathcal{V}, \mathcal{Q})$ and an admissible economy $P \in \mathcal{D}^I$, the dynamic TTC rule specifies an allocation by implementing the algorithm below. First, we check label of the root $\mathcal{V}(v^0)$ and let it be the first subset of trading houses $\hat{H}_1$. Implement the TTC restricted to $\hat{H}_1$, and denote the resulting allocation as $m_1 = TTC(P, e, \hat{H}_1)$. Then we identify the unique arc $(v^0, v) \in Q_{v^0}$ such that $\mathcal{Q}(v^0, v) = m_1$. Next, let the second subset of trading houses be the label of v, i.e., $\hat{H}_2 = \mathcal{V}(v)$ and implement $TTC(P, m_1, \hat{H}_2)$. We repeatedly do this until we arrive at a terminal vertex. Formally

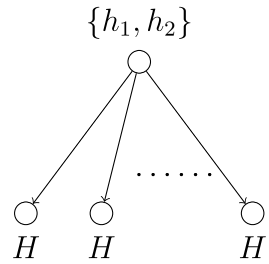
Definition 5. *Fix a trading tree T,* ***a dynamic TTC (DTTC) rule*** *is a mapping $\varphi^T : \mathcal{D}^I \to M$ which specifies for each admissible economy P an allocation $\varphi^T(P)$ by following steps*

- *Let $v_1 = v^0$, $\hat{H}_1 = \mathcal{V}(v_1)$ and $m_1 = TTC(P, e, \hat{H}_1)$.*
- *For each $k = 2, \cdots, K$, let $v_k \in V$ be such that $\mathcal{Q}(v_{k-1}, v_k) = m_{k-1}$, $\hat{H}_k = \mathcal{V}(v_k)$ and $m_k = TTC(P, m_{k-1}, \hat{H}_k)$, where K is uniquely identified by v_K being a terminal vertex.*

Several examples of DTTC rules are presented below.

Example 3. Let $T = (V, Q, \mathcal{V}, \mathcal{Q})$ where $V = \{v^0\}$ and $Q = \emptyset$. By condition C1, $\mathcal{V}(v^0) = H$. Then the DTTC rule defined accordingly is exactly Gale's TTC rule.

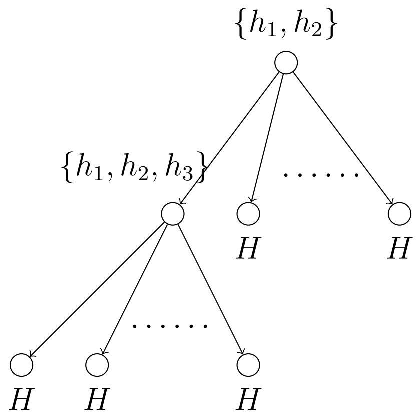
Example 4. Let $T = (V, Q, \mathcal{V}, \mathcal{Q})$ where $V = \{v^0\} \cup \{v_k : k = 1, \cdots, n!\}$ and $Q = \{(v^0, v_k) : k = 1, \cdots, n!\}$. The unique non-terminal vertex is v^0 and $Z = \{v_k : k = 1, \cdots, n!\}$. Label the vertices such that $\mathcal{V}(v^0) = \{h_1, h_2\}$ and $\mathcal{V}(v_k) = H$ for each $v_k \in Z$. In addition, label the arcs by an arbitrary one-to-one mapping form Q to M.



The accordingly defined DTTC rule first gives agents 1 and 2 an opportunity to exchange their houses. After that Gale's TTC rule is implemented upon the resulting allocation. This is the formal representation of the first DTTC rule we discussed in Example 1.

Example 5. Let $T = (V, Q, \mathcal{V}, \mathcal{Q})$ where $V = \{v^0\} \cup \{v_k^1 : k = 1, \cdots, n!\} \cup \{v_k^2 : k = 1, \cdots, n!\}$ and $Q = \{(v^0, v_k^1) : k = 1, \cdots, n!\} \cup \{(v_1^1, v_k^2) : k = 1, \cdots, n!\}$. In this case, there are two non-terminal vertices, v^0 and v_1^1. The set of terminal vertices is $Z = V \backslash \{v^0, v_1^1\}$. Label the vertices such that $\mathcal{V}(v^0) = \{h_1, h_2\}$, $\mathcal{V}(v_1^1) = \{h_1, h_2, h_3\}$, and $\mathcal{V}(v) = H$ for all $v \in Z$. In addition label the arcs Q_{v^0} by an one-to-one mapping from Q_{v^0} to M such that

$$\mathcal{Q}(v^0, v_1^1) = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ h_2 & h_1 & h_3 & \cdots & h_n \end{pmatrix}.$$



Finally label the remaining arcs by an arbitrary one-to-one mapping from $Q_{v_1^1}$ to M.

The dynamic top trading cycles rule defined accordingly gives first agents 1 and 2 an opportunity to exchange their houses. The next subset of trading houses depends on whether they really exchanged their houses. If so, the next subset of trading houses is $\{h_1, h_2, h_3\}$, followed by Gale's TTC rule. And if otherwise, Gale's TTC is implemented directly. This is the formal representation of the second DTTC rule in Example 1.

Remark 1. Notice that C2 requires that from each vertex, one arc is presented for each allocation in M. In our analysis, we fix the endowment $e \in M$ hence there are many redundant paths in the tree. For example in the tree presented in Example 4, the only possible changes from the endowment after the first-step TTC is that agent 1 and 2 exchanged their houses, which means only two arcs are reachable from v^0. However we specify for each one of $n!$ allocations an arc. We reserve these redundancies deliberately in order to make a DTTC rule applicable whatever the endowment is.

From the definition, it is evident that whenever an agent exchanges her house with some other agent, she gets strictly better off. Hence a DTTC rule is individually rational. In addition, since every path in a trading tree terminates with the full set H, every possible Pareto improvement has been exploited and hence a DTTC rule is efficient. The proposition below presents these two observations without a proof.

Proposition 1. *A DTTC rule is efficient and individually rational.*

A specially interesting sub-class of DTTC rules is the collection of those defined by degenerated trading trees. A trading tree is degenerated if two vertices are labeled the same as long as the distances away from the root are the same. Formally

Definition 6. *A trading tree $T = (V, Q, \mathcal{V}, \mathcal{Q})$ is* ***degenerated*** *if for all $v, v' \in V$,*

$$|P(v^0, v)| = |P(v^0, v')| \Rightarrow \mathcal{V}(v) = \mathcal{V}(v').$$

A degenerated trading tree can be conveniently expressed as a sequence of subsets of trading houses. For example[5] $S = (\hat{H}_1, \hat{H}_2, \cdots, \hat{H}_K)$ indicates that the first subset of trading houses is

[5]To distinguish a degenerated tree from those are not, we label it by S rather than T.

$\hat{H}_1$. After that, no matter what $TTC(P, e, \hat{H}_1)$ is, the next subset of trading houses will be $\hat{H}_2$. And so on.

We call a DTTC rule defined by a degenerated trading tree a **sequential TTC (STTC) rule.** The Gale's TTC rule is an STTC rule, which is defined by the sequence $S = (H)$. In addition the DTTC rule described in Example 4 is also an STTC rule, which is defined by the sequence $S = (\{h_1, h_2\}, H)$. However, the DTTC rule in Example 5 is not an STTC rule. Another example of STTC rules is in Example 6.

Example 6. Bade (2017) recently proposed a strategy-proof mechanism for the house exchange problem with single-peaked preferences, called "crawler." This mechanism is implemented step by step. In each step, we identify the first agent, from the one holding the smallest house to the one holding the largest, whose favorite house is no larger than her currently holding house. If her favorite is exactly the house she holds, let it be her final allocation. Otherwise, let the agent be i and let her favorite house be h_j. Then $h_j < h_i$, where h_i denotes agent i's currently holding house. Let h_j be agent i's final allocation and let every agent, whose currently holding house is larger than or equal to h_j and smaller than h_i, "crawl" to the house slightly larger. Exactly one agent gets her final allocation in each step and hence the mechanism terminates with n steps.

The "crawler" appears very different from TTC. However, it is equivalent to an STTC defined by the degenerated trading tree below

$$S = (\{h_1, h_2\}, \{h_2, h_3\}, \{h_1, h_2\}, \{h_3, h_4\}, \{h_2, h_3\}, \{h_1, h_2\}, \cdots).$$

In particular, the first subset of trading houses is $\{h_1, h_2\}$. The next two are $\{h_2, h_3\}$ and $\{h_1, h_2\}$. The next three are $\{h_3, h_4\}$, $\{h_2, h_3\}$, $\{h_1, h_2\}$. And so on. The reader is encouraged to apply this STTC rule to the Example 1 in Bade (2017) to see the equivalence.

4 Strategy-Proofness and Single-Peakedness

In this section we examine the implication of strategy-proofness on the choices of DTTC rules. Such an implication depends clearly on the preference domain. If all preferences are admissible, i.e., the universal domain is the admissible domain, then the class of DTTC rules collapses to Gale's TTC.

Proposition 2. *A DTTC rule $\varphi^T : \mathcal{P}^I \to M$ is strategy-proof iff it is Gale's TTC rule.*

This is a direct implication of Proposition 1 and Ma (1994)'s characterization which proves that a rule on the universal domain is strategy-proof, efficient, and individually rational if and only if it is Gale's TTC.

However, if the admissible preferences are single-peaked, we identify a sub-class of DTTC rules to be strategy-proof. Single-peakedness for the house exchange problems may be defined with respect to the size of houses or the market value of them. For the following analysis, we assume that houses are ordered according to size. Such a linear order is denoted as $<$ and without loss of generality we fix it as $h_1 < h_2 < \cdots < h_n$ throughout the analysis. As we interpret this order as according to size, we will say h is smaller than h' if $h < h'$. In addition we denote $h \leqslant h'$ if either h is smaller than h' or they are identical. We assume that each agent's preference is single-peaked with respect to $<$. Formally,

Definition 7. *A strict preference P_i on H is **single-peaked** with respect to $<$ if*

$$h' < h \leqslant \tau(P_i) \Rightarrow h\ P_i\ h' \text{ and } \tau(P_i) \leqslant h < h' \Rightarrow h\ P_i\ h'.$$

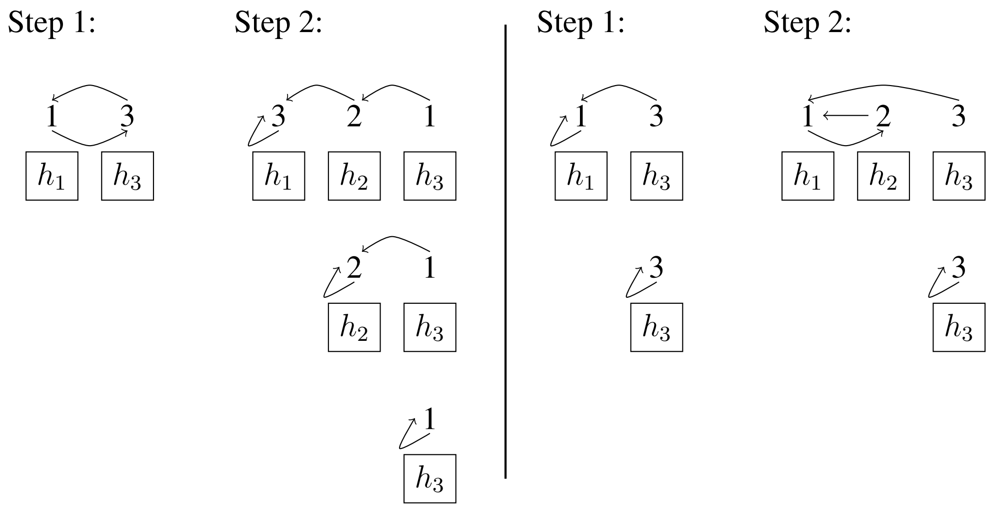
Let $\mathcal{SP}_<$ denote the collection of all single-peaked preferences. In the remaining of this section, we study DTTC rules $\varphi^T : \mathcal{SP}^I_< \to M$. The example below shows that such a rule is generally manipulable. However, it at the same time suggests a restriction on the trading tree, which is later shown sufficient to guarantee strategy-proofness.

Example 7. Let $I = \{1, 2, 3\}$, $H = \{h_1, h_2, h_3\}$, and P be as below.

$$P = \begin{pmatrix} P_1: & h_2 & h_3 & h_1 \\ P_2: & h_1 & h_2 & h_3 \\ P_3: & h_1 & h_2 & h_3 \end{pmatrix}$$

Let $P'_1 : h_2\ h_1\ h_3$ and $P' = (P'_1, P_{-1})$.

Consider an STTC rule defined by the degenerated tree $S = (\{h_1, h_3\}, H)$. Then the dynamic TTC procedures for the two preference profiles are illustrated below.



At P, in the first step, agents 1 and 3 will exchange their houses. And in the second step, agent 3 will point to herself while agent 2 points to her and agent 1 points to agent 2. Then the unique cycle involves agent 3 herself. In the next round of TTC agent 2 points to herself and finally agent 1 points to herself. So $\varphi_1^S(P) = h_3$.

However, if agent 1 deviates to P'_1. In the first step, she will no longer exchange with agent 3 and stay with her own house. Then in the second step she can get h_2, which is better than h_3 at P.

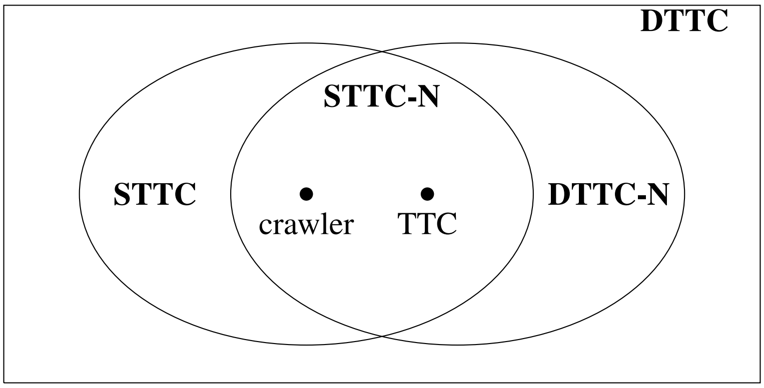
The above example suggests that in any step, the subset of trading houses is a neighborhood. We now formalize this notion.

Definition 8. *A nonempty subset of houses $\hat{H} \subset 2^H \backslash \emptyset$ is **a neighborhood** if*

$$\forall\, h, h', h'' \textit{ such that } h < h'' < h',\; h, h' \in \hat{H} \Rightarrow h'' \in \hat{H}.$$

Let $\mathcal{H}$ denote the collection of all neighborhoods. A trading tree $T = (V, Q, \mathcal{V}, \mathcal{Q})$ is a **neighborhood tree** if $\mathcal{V}(v) \in \mathcal{H}$ for all $v \in V$. A DTTC rule is **a dynamic TTC rule with neighborhoods (DTTC-N rule)** if it is defined by a neighborhood tree.

If a DTTC-N rule is also a sequential TTC rule, we denote it as STTC-N. Gale's TTC rule, the DTTC rules in Example 4 and 5, and the "crawler" of Bade (2017) are all STTC-N rules. The following Venn diagram shows the relations among various classes of DTTC rules.



Below is a main result, which states that any DTTC defined by a neighborhood tree is strategy-proof on the single-peaked domain.

Theorem 1. *A DTTC-N rule* $\varphi^T : \mathcal{SP}^I_< \to M$ *is strategy-proof.*

The proof is in Appendix A.

We now address the question: If an STTC rule is strategy-proof on the single-peaked domain, does it need to be an STTC-N rule? The answer is yes, up to outcome-equivalence. In particular, we prove that whenever an STTC rule is strategy-proof on the single-peaked domain, it is equivalent to an STTC-N rule in the sense that they always select the same allocation for all admissible economies.

In the construction of the STTC-N rule which is equivalent to the given STTC rule, we define the closure of a subset of trading houses. To be precise, for an arbitrary nonempty subset of houses $\hat{H} \subset 2^H \backslash \emptyset$, let $cl(\hat{H})$ be the smallest neighborhood that contains it, i.e., $cl(\hat{H}) = \min_{\subset}\{\tilde{H} \in \mathcal{H} : \hat{H} \subset \tilde{H}\}$. We call $cl(\hat{H})$ **the closure** of $\hat{H}$. For example, $cl(\{h_1, h_3\}) = \{h_1, h_2, h_3\}$. In addition, the closure of a neighborhood is itself.

Fix an arbitrary degenerated trading tree $S = \left(\hat{H}_1, \hat{H}_2, \cdots, \hat{H}_K\right)$, let

$$cl(S) \equiv \left(cl(\hat{H}_1), cl(\hat{H}_2), \cdots, cl(\hat{H}_K)\right).$$

It is evident that for any degenerated trading tree S, $cl(S)$ is a neighborhood tree.

The theorem below shows that whenever an STTC rule φ^S is strategy-proof on the single-peaked domain, it selects an allocation that is the same as selected by the STTC-N rule $\varphi^{cl(S)}$ for every admissible economy.

Theorem 2. *If an STTC rule* $\varphi^S : \mathcal{SP}^I_< \to M$ *is strategy-proof, then for all* $P \in \mathcal{SP}^I_<$, $\varphi^S(P) = \varphi^{cl(S)}(P)$.

The proof is in Appendix B.

Finally, we address the question: How can we expand the single-peaked domain while reserving the strategy-proofness of an arbitrary DTTC-N rule? The answer is definitely in negative. We prove that whenever the single-peaked domain is expanded by adding even just one non-single-peaked preference, there is an DTTC-N rule which is manipulable.

Theorem 3. *Let* $P_0 \in \mathcal{P} \backslash \mathcal{SP}_<$ *be arbitrary. There is a degenerated tree of neighborhoods* S *such that* $\varphi^S : \{\mathcal{SP}_< \cup \{P_0\}\} \to M$ *is manipulable.*

The proof is in Appendix C.

5 Conclusion

This paper presents one of the first investigations on the designing problem of strategy-proof exchange rules on the single-peaked domain. By identifying a large class of such rules, we believe that the mechanism designer is now endowed with much freedom to cope with context-specific requirements.

However, there are questions unsolved. Probably the most interesting is on the existence of a rule witch is not a DTTC-N rule but still satisfies all three axioms. This is a difficult question and deserves further study.

Appendix

A Proof of Theorem 1

Pick an arbitrary preference profile $P \in \mathcal{SP}^I_<$ and let an arbitrary agent i be the unilateral deviater. Without loss of generality, we assume that agent i's favorite house is larger than her endowment, i.e., $\tau(P_i) > h_i$. We track agent i's holding houses in the procedure of DTTC

$$h_i = h^1 \to h^2 \to \cdots \to h^q \to h^{q+1} \to \cdots \to h^{Q-1} \to h^Q = \varphi_i^T(P).$$

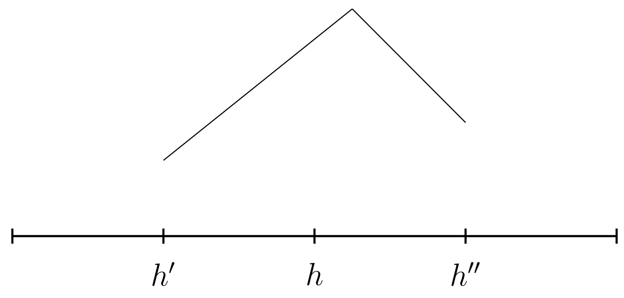
From the beginning, agent i's holding house is h_i, which is denoted h^1. Then in the first step when h^1 is allowed to be traded, agent i changes her holding house from h^1 to h^2.[6] We keep track of the changes of her holding houses in all steps when her holding house is allowed to be traded, until her final allocation $\varphi_i^T(P)$, which is denoted as h^Q.

In order to prove strategy-proofness, we show a series of lemmas which depict the above sequence and then the implication of her unilateral deviation to the sequence. The first two lemmas study the question when will an agent stops exchanging houses.

[6]Notice that h^1 might be the same as h^2. Hence although she is allowed to trade, her holding house is not changing.

Lemma 1. *In any step k, any round t of the TTC in that step, and any agent $i \in C_t$ involved in a cycle in that round such that agent i gets a house which is in between two other houses which are available in that round, i.e., let $TTC_i(P, m_{k-1}, \hat{I}_k) = h$ and there are $h', h'' \in \tilde{H}_t$ such that $h' < h < h''$. Agent i will not trade again in any future step, i.e., $\varphi_i^T(P) = h$.*

Proof. The picture below illustrates the situation studied by the lemma.



From single-peakedness and the fact that agent i points to h when h' and h'' are still available, for every h_j s.t. $h_j \; P_i \; h$, $h' < h_j < h''$ and h_j has been taken by some other agent, say j. Then to show $\varphi_i^T(P) = h$, it suffices to show every such agent j will not exchange again in any future step. To do this, we pick an arbitrary such agent j, and apply the same logic we applied to agent i. (Notice that when agent j takes h_j, the houses h' and h'' are also available.) Then to show that agent j will not trade again, it suffices to show that some agent k will not trade again.

Since the houses in between h' and h'' are finite, there must be a group of agents, each of who gets a house in the first round in the TTC in step k. In addition, by domino effect, agent i will not trade again in any future step if no one in this group will trade again, which is evident by the fact that every house in between h' and h'' is available for trade and single-peakedness. □

Lemma 2. *Let in a certain step agent i changes her holding house from h^q to h^{q+1} such that $h^q < \tau(P_i) \leqslant h^{q+1}$, then $\varphi_i^T(P) = h^{q+1}$.*

Proof. If $h^{q+1} = \tau(P_i)$, the conclusion is evident. We prove the lemma for the case where $\tau(P_i) < h^{q+1}$. By the fact that the trading houses form a neighborhood, every house that is better than h^{q+1} is available for exchange in this step. Then agent i gets h^{q+1} from the TTC in this step means, in the round when she gets h^{q+1}, every house h that is better than h^{q+1} has been taken by some other agent in some previous round. In other words, every such house h is taken by someone in a round when h^q and h^{q+1} are still available in this round and $h^q < h < h^{q+1}$. Then Lemma 1 implies that the agent who took h will never exchange again. This is true for every house that is better than h^{q+1} according to P_i. Hence agent i will not exchange in any future step either, i.e., $\varphi_i^T(P) = h^{q+1}$. □

The third lemma studies the sequence of agent i's holding houses when her final allocation is smaller than or equal to her peak. It says that throughout the procedure agent i never hold a house larger than her final allocation. The situation is illustrated in Figure 1.

Lemma 3. *If $\varphi_i^T(P) \leqslant \tau(P_i)$, then $h^1 \leqslant h^2 \leqslant \cdots \leqslant h^Q = \varphi_i^T(P)$.*

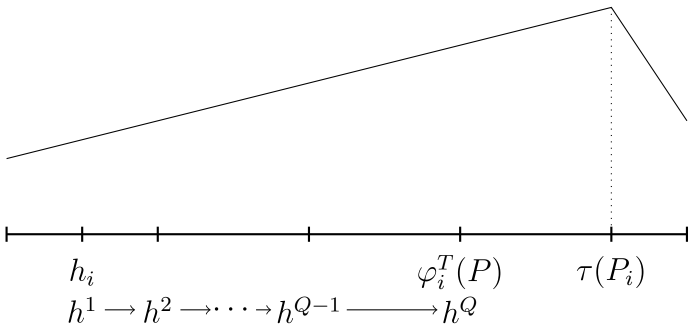


Figure 1: The sequence of agent i's holding houses when $\varphi_i^T(P) \leqslant \tau(P_i)$.

Proof. By individual rationality of TTC and the fact that $\varphi_i^T(P) \leqslant \tau(P_i)$, it suffices to show $h^2, \cdots, h^{Q-1} \leqslant \varphi_i^T(P)$. Suppose not, let $\bar{q} = \min_q\{h^q : q = 2, \cdots, Q-1 : h^q > \varphi_i^T(P)\}$. It is either $\varphi_i^T(P) < h^{\bar{q}} < \tau(P_i)$ or $h^{\bar{q}} \geqslant \tau(P_i)$. The former case can not happen because whenever agent holds $h^{\bar{q}}$, she will never exchange it for $\varphi_i^T(P)$. The latter implies $\varphi_i^T(P) = h^{\bar{q}}$ according to Lemma 2: contradiction. □

The next lemma studies the sequence of agent i's holding houses when her final allocation is larger than her peak. It says that agent i changes her holding house from one smaller than $\tau(P_i)$ to one larger than $\tau(P_i)$ in a certain step, and she will not exchange any more. The situation is illustrated in Figure 2.

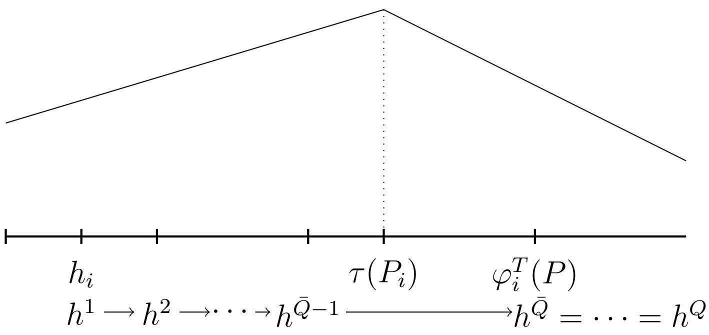


Figure 2: The sequence of agent i's holding houses when $\varphi_i^T(P) > \tau(P_i)$.

Lemma 4. *If $\varphi_i^T(P) > \tau(P_i)$, then there is unique $\bar{Q} \in \{2, \cdots, Q\}$ such that*

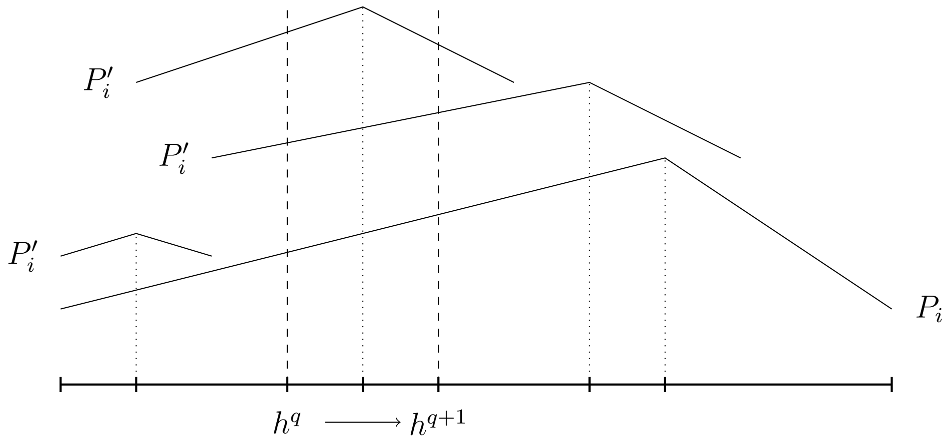
$$h^1 \leqslant h^2 \leqslant \cdots < h^{\bar{Q}-1} < \tau(P_i) < h^{\bar{Q}} = \cdots = h^Q.$$

Proof. This lemma is directly implied by Lemma 2 and the fact that, whenever agent i changes her holding house from h^q to h^{q+1}, $h^{q+1}\ P_i\ h^q$. □

Given agent i's sequence of holding houses, we investigate the changes due to her unilateral deviation from P_i to an arbitrary $P'_i \in \mathcal{SP}_<$. We identify the first divergence between the sequence of holding houses at P and the sequence at (P'_i, P_{-i}). In particular, let it be such that agent i changes her holding house from h^q to h^{q+1} at P and to h^* at (P'_i, P_{-i}). By assumption $h^* \neq h^{q+1}$.

With following lemmas, we prove $h^{q+1}\ P_i\ \varphi_i^T(P'_i, P_{-i})$, which implies $\varphi_i^T(P)\ P_i\ \varphi_i^T(P'_i, P_{-i})$. In particular, Lemma 5 deals with the situation where $h^q < h^{q+1} \leqslant \tau(P_i)$, Lemma 6 deals with the situation where $h^q = h^{q+1} \leqslant \tau(P_i)$, Lemma 7 deals with the situation where $h^q < \tau(P_i) < h^{q+1}$, and finally Lemma 8 deals with the situation where $h^q = h^{q+1} > \tau(P_i)$.

Lemma 5. *If $h^q < h^{q+1} \leqslant \tau(P_i)$, then $h^{q+1}\ P_i\ \varphi_i^T(P'_i, P_{-i})$.*



Proof. **Case 1:** $\tau(P'_i) \geqslant h^{q+1}$. Let $\underline{h}$ and $\overline{h}$ denote respectively the smallest and largest houses in the trading houses in this step. Then every house in between is allowed to be traded. We discuss the following four sub-cases.

If $\overline{h} \leqslant \min_<\{\tau(P_i), \tau(P'_i)\}$, then the induced preferences on the trading houses of P_i and P'_i are the same, which implies $h^* = h^{q+1}$: contradiction.

If $\tau(P_i) < \overline{h} \leqslant \tau(P'_i)$, by strategy-proofness and individual rationality of TTC, $h^* \neq h^{q+1}$ implies $h^{q+1}\ P_i\ h^*$ and hence it is either that $h^q \leqslant h^* < h^{q+1}$ or that $\tau(P_i) < h^*$ and $h^{q+1}\ P_i\ h^*$. The former is impossible by the definition of TTC and the facts that (i) agent i will never point to such an h^* whenever h^{q+1} is available and (ii) that the induced preferences on $[h^q, h^{q+1}]$ of P_i and P'_i are the same. For the latter case, no matter what are the further exchanges for agent i, by $\tau(P_i) < h^* \leqslant \overline{h} \leqslant \tau(P'_i)$ and our analysis in Lemma 3 and 4, $\tau(P_i) < h^* \leqslant \varphi_i^T(P'_i, P_{-i})$ and hence $h^{q+1}\ P_i\ \varphi_i^T(P'_i, P_{-i})$.

If $\tau(P'_i) < \overline{h} \leqslant \tau(P_i)$, by strategy-proofness and individual rationality of TTC, $h^* \neq h^{q+1}$ implies $h^{q+1}\ P_i\ h^*$ and hence it is either that $h^q \leqslant h^* < h^{q+1}$ or that $\tau(P_i) < h^*$ and $h^{q+1}\ P_i\ h^*$. By the assumption that $\overline{h} \leqslant \tau(P_i)$, the latter is impossible. In addition, the former case is ruled out by the same argument in the above paragraph.

If $\overline{h} > \max_<\{\tau(P_i), \tau(P'_i)\}$, an argument similar to the above proves what we want.

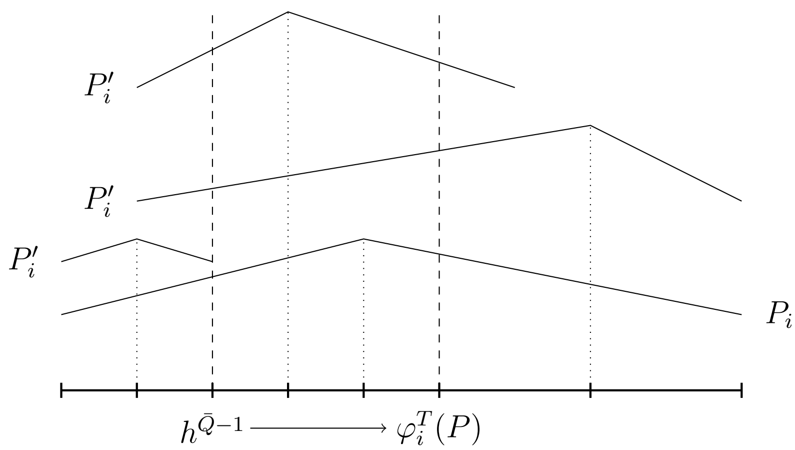
Case 2: $h^q < \tau(P'_i) < h^{q+1}$. Strategy-proofness and individual rationality of TTC imply either $h^q \leqslant h^* < h^{q+1}$ or $h^* > \tau(P_i)$ and $h^{q+1}\ P_i\ h^*$. In the latter case, $h^* > \tau(P'_i)$ and hence Lemma 2 implies $\varphi_i^T(P'_i, P_{-i}) = h^*$, which in turn implies $h^{q+1}\ P_i\ \varphi_i^T(P'_i, P_{-i})$. So in the remaining, $h^q \leqslant h^* < h^{q+1}$. Notice that at P, agent i changes from h^q to h^{q+1}. Hence there is a cycle of agents who form a cycle with agent i in a certain step. In addition, at (P_i', P_{-i}), these agent still do the same pointing as long as agent i hasn't get a house. In particular, the agent who owns h^{q+1} will not get a house before agent i at (P_i', P_{-i}). Consequently, if $h^q < h^* < h^{q+1}$, Lemma 1 implies $\varphi_i^T(P_i', P_{-i})$ and hence $h^{q+1}\ P_i\ \varphi_i^T(P_i', P_{-i})$. If otherwise, $h^* = h^q$. Every house h such that $h\ P_i'\ h^q$ has been taken by some other before agent i gets h^q. Then Lemma 1 implies that those agents will not exchange again and hence $\varphi_i^T(P_i', P_{-i}) = h^q$.

Case 3: $\tau(P_i') \leqslant h^q$. The conclusion is evidently true. □

Lemma 6. *If* $h^q = h^{q+1} \leqslant \tau(P_i)$, *then* $h^{q+1}\ P_i\ \varphi_i^T(P_i', P_{-i})$.

Proof. If $\tau(P_i') \leqslant h^q$, the conclusion is trivial. We assume $\tau(P_i') > h^q$. By strategy-proofness and individual rationality of TTC, $h^* \neq h^{q+1}$ implies that h^* is on the other side of the peak, i.e., $h^* > \tau(P_i)$, and $h^{q+1}\ P_i\ h^*$. In addition, Lemma 3 and 4 imply $h^* \leqslant \varphi_i^T(P_i', P_{-i})$ and hence $h^{q+1}\ P_i\ \varphi_i^T(P_i', P_{-i})$. □

Lemma 7. *If* $h^q < \tau(P_i) < h^{q+1}$, *then* $h^{q+1}\ P_i\ \varphi_i^T(P_i', P_{-i})$.



Proof. Notice first $q = \bar{Q} - 1$ and $h^{q+1} = \varphi_i^T(P)$.

Case 1: $\tau(P_i') \geqslant \varphi_i^T(P)$. By strategy-proofness and individual rationality of TTC, $h^* \neq \varphi_i^T(P)$ implies either (i) $h^* < \tau(P_i)$ and $\varphi_i^T(P)\ P_i\ h^*$ or (ii) $\varphi_i^T(P) < h^*$. The former is impossible because according to P_i', agent i will not point to h^* before $\varphi_i^T(P)$ is taken by some others. In addition, since at P agent i gets $\varphi_i^T(P)$, there is a group of agents who form a cycle with agent i. At (P_i', P_{-i}), there will still do the same pointing as long as agent i hasn't taken a house. As to the latter, Lemma 3 and 4 imply $\varphi_i^T(P_i', P_{-i}) \geqslant h^*$ and hence $\varphi_i^T(P)\ P_i\ \varphi_i^T(P_i', P_{-i})$.

Case 2: $h^{Q-1} < \tau(P_i') < \varphi_i^T(P)$. For this case, the same logic as in the proof of Case 2, Lemma 5 proves the conclusion.

Case 3: $\tau(P_i') \leqslant h^{Q-1}$. The conclusion is trivially true. □

Lemma 8. *If* $h^q = h^{q+1} > \tau(P_i)$, *then* $h^{q+1}\ P_i\ \varphi_i^T(P_i', P_{-i})$.

Lemma 8 can be proved with arguments similar as in the proof of Lemma 6.

Now we are ready to claim the theorem. According to the above four lemmas, in the first step when agent i is allowed to trade, if her holding house changes from h^1 to h^2 (which might be the same as h^1) at P but to some h^* that is different with h^2 at (P_i', P_{-i}), then her finally house at (P_i', P_{-i}) will be worse than h^2. Hence if P_i' is a profitable manipulation at P, $h^* = h^2$.

After that, we consider the next step when agent i is allowed to trade. Due to the tree structure where the next subset of trading houses depends only on the allocation from the previous step, the second step when agent i is allowed to trade at (P_i', P_{-i}) is the same as at P. Then the logic we apply to the first step when i is allowed to trade applies also to the second step. Hence, after we consider all the steps when agent i is allowed to trade, we conclude that if P_i' is a profitable manipulation at P, her trace of holding houses at (P_i', P_{-i}) is the same as at P, which is contradicting the assumption of profitable manipulation.

B Proof of Theorem 2

If S is a neighborhood tree, the result is trivial. Let, without loss of generality, $\hat{H}_{\bar{k}} \notin \mathcal{H}$ be the first non-neighborhood subset of trading houses along the sequence S. Let

$$\bar{S} = \left(\hat{H}_1, \cdots, \hat{H}_{\bar{k}-1}, cl(\hat{H}_{\bar{k}}), \hat{H}_{\bar{k}+1}, \cdots, \hat{H}_K\right)$$

be a degenerated tree whose only difference from S is the replacement of $\hat{H}_{\bar{k}}$ by $cl(\hat{H}_{\bar{k}})$. Hence to show the theorem, it suffice to show that $\varphi^S : \mathcal{SP}^I_< \to M$ being strategy-proof implies $\varphi^S(P) = \varphi^{\bar{S}}(P)$ for all $P \in \mathcal{SP}^I_<$, which is equivalent to showing that the allocation after the $\bar{k}$-th step TTC of $\varphi^S(P)$ and $\varphi^{\bar{S}}(P)$ are the same.

In addition, although $\hat{H}_{\bar{k}}$ is generally a union of more than two neighborhoods, we assume that it consists of exactly two neighborhoods. We can do this because as long as we can show the above statement for such an $\hat{H}_{\bar{k}}$, the result can be easily generalized to the general cases. Formally, let $\hat{H}^1, \hat{H}^2, \hat{H}^3 \in \mathcal{H}$ be three neighborhoods such that $h^1 < h^2 < h^3$ for arbitrary $h^1 \in \hat{H}^1$, $h^2 \in \hat{H}^2$, and $h^3 \in \hat{H}^3$. Let $\hat{H}_{\bar{k}} = \hat{H}^1 \cup \hat{H}^3 \notin \mathcal{H}$ and $\hat{H}^1 \cup \hat{H}^2 \cup \hat{H}^3 \in \mathcal{H}$. For convenience, we denote $h < \hat{H}^1$ if $h < h'$ for all $h' \in \hat{H}^1$.

The following two lemmas are implications of the strategy-proofness of $\varphi^S : \mathcal{SP}^I_< \to M$.

Lemma 9. *There are $k, k' < \bar{k}$ such that (i) $\hat{H}^1 \cup \hat{H}^2 \subset \hat{H}_k$ and (ii) $\hat{H}^2 \cup \hat{H}^3 \subset \hat{H}_{k'}$.*

Proof. We show $\hat{H}^1 \cup \hat{H}^2 \subset \hat{H}_k$ and the other can be shown symmetrically. Suppose not, i.e., $\nexists k < \bar{k}$ s.t. $\hat{H}^1 \cup \hat{H}^2 \subset \hat{H}_k$. Then let $h^1, h^2, h^3 \in H$ be houses such that $h^1 = \min_< H^1$, $h^2 = \max_< H^2$, and $h^3 \in H^3$. So h^1 is the smallest house in H^1, h^2 is the largest house in H^2, and h^3 is arbitrary in H^3. Let i^1, i^2, i^3 be agents whose endowments are respectively h^1, h^2, h^3, i.e., $e(i^1) = h^1$, $e(i^2) = h^2$, and $e(i^3) = h^3$.

Consider a preference profile $P \in \mathcal{SP}^I_<$ such that $h^2 \; P_{i^1} \; h^3 \; P_{i^1} \; h^1$, $h^1 \; P_{i^2} \; h^2 \; P_{i^2} \; h^3$, $h^1 \; P_{i^3} \; h^2 \; P_{i^3} \; h^3$, and for all $i \neq i^1, i^2, i^3$, $\tau(P_i) = e(i)$. We apply φ^S to P. Every agent other than i^1, i^2, i^3 is endowed her favorite house and hence will never exchange with anybody. In addition, the first exchange along the steps of the algorithm must be either between i^1 and

i^2 or between i^1 and i^3. Hence there will be no exchange until a subset of trading houses including either h^1, h^2 or h^1, h^3. So the first exchange will be i^1 and i^3 exchanging their houses in the $\bar{k}$-th step. After this, agent i^3 gets her favorite house among h^1, h^2, h^3 and hence will not exchange again. Agents i^1 and i^2's preferences between h^2 and h^3 are the same and hence will not exchange either. So $\varphi^S_{i^1}(P) = h^3$.

Consider that agent i^1 reports another preference $P'_{i^1} \in \mathcal{SP}_<$ such that $h^2 \ P'_{i^1} \ h^1 \ P'_{i^1} \ h^3$. We apply φ^S to (P'_{i^1}, P_{-i^1}). By such a deviation, agent i^1 and i^3 will no longer exchange their houses in the $\bar{k}$-th step. After this step, by the definition of a trading tree, there is $k'' > \bar{k}$ such that $\hat{H}^1 \cup \hat{H}^2 \subset \hat{H}_{k''}$. Without loss of generality, let $\hat{H}_{k''}$ be the first such subset of trading houses along S. Then agent i^1 and i^2 will exchange their houses in step k'', and it is evident that this is the last step when there is some exchange. Hence $\varphi^S_{i^1}(P'_{i^1}, P_{-i^1}) = h^2 \ P_{i^1} \ h^3 = \varphi^S_{i^1}(P)$: manipulation! □

For all $h < h' \in H$, **h′ is reachable from h in k steps** if $\exists$ a subsequence $(\hat{H}_{k_m})_{m=1}^M \subset (\hat{H}_1, \cdots, \hat{H}_k)$ such that (i) $h \in \hat{H}_{k_1}$, (ii) $\hat{H}_{k_m} \cap \hat{H}_{k_{m+1}} \neq \emptyset \ \forall m = 1, \cdots, M-1$, and (iii) $h' \in \hat{H}_{k_M}$. It is evident that h' is reachable from h if and only if an agent i such that $e(i) = h$ gets h' in k steps at the economy $P \in \mathcal{SP}^I_<$ such that (i) $\tau(P_i) = h'$, (ii) $\tau(P_j) = h_1$ for all $j \in I$ s.t. $h < e(j) \leqslant h'$, and (iii) $\tau(P_l) = e(l)$ for all the other agents. In addition, for any $h < h' < h''$, if h'' is reachable from h in k steps, by definition, h' is also reachable from h in k steps. We omit the symmetric definition of reachability from h to h' in k steps where $h' < h$.

Lemma 9 implies that an arbitrary $h^2 \in \hat{H}^2$ is reachable from an arbitrary $h^1 \in \hat{H}^1$ in $\bar{k}-1$ steps and an arbitrary $h^2 \in \hat{H}^2$ is also reachable from an arbitrary $h^3 \in \hat{H}^3$ in $\bar{k}-1$ steps. In addition to reachability from $\hat{H}^1$ and $\hat{H}^3$ to $\hat{H}^2$, we have also the following lemma.

Lemma 10. *The following two statements are true.*

1. *For arbitrary $h < \hat{H}^1$, if $\exists h^1 \in \hat{H}^1$ which is reachable from h in $\bar{k}-1$ steps, then every $h^2 \in \hat{H}^2$ is also reachable from h in $\bar{k}-1$ steps.*

2. *For arbitrary $h > \hat{H}^3$, if $\exists h^3 \in \hat{H}^3$ which is reachable from h in $\bar{k}-1$ steps, then every $h^2 \in \hat{H}^2$ is also reachable from h in $\bar{k}-1$ steps.*

Proof. We prove the first statement and the second can be proved symmetrically. Suppose the first statement is not true, without loss of generality, let h^1 be the largest among those in $\hat{H}^1$ and are reachable from h in $\bar{k}-1$ steps, let h^2 be the smallest among those in $\hat{H}^2$ and are not reachable from h in $\bar{k}-1$ steps, and let $h^3 \in \hat{H}^3$ be arbitrary. Let in addition $e(i) = h$, $e(i^1) = h^1$, $e(i^2) = h^2$, and $e(i^3) = h^3$.

Consider a preference profile $P \in \mathcal{SP}^I_<$ where (i) $h^2 \ P_i \ h^3 \ P_i \ h^1 \ P_i \ h$, (ii) $\tau(P_j) = h_1$ for all $j \in \{l \in I : h < e(l) < h^1\} \cup \{i^1, i^2, i^3\}$, and $\tau(P_l) = e(l)$ for all other agents. We apply φ^S to P. Since h^1 is reachable from h in $\bar{k}-1$ steps and h^2 is not reachable from h in $\bar{k}-1$ steps, agent i holds h^1 after the $(\bar{k}-1)$-th step of TTC. Then in the $\bar{k}$-th step, agent i will exchange h^1 with i^3 for h^3. After that no exchange will happen and hence $\varphi^S_i(P) = h^3$.

However, if agent i misreports a preference P'_i such that $h^2 \ P'_i \ h^1 \ P'_i \ h^3 \ P'_i \ h$. In the procedure of applying φ^S to (P'_i, P_{-i}), agent i still holds h^1 after the $(\bar{k}-1)$-th step but she will not exchange h^1 with agent i^3 for h^3 in the $\bar{k}$-th any more. Also, by the definition of a trading tree, agent i will exchange with i^2 in some later step and hence $\varphi_i^S(P_i', P_{-i}) = h^2$ which is strictly better than h^3: manipulation! $\square$

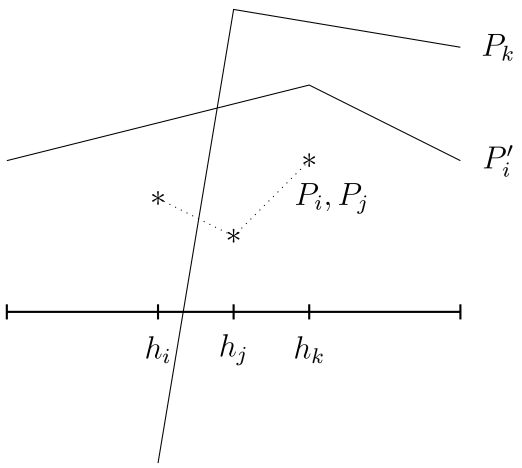
We are now ready to claim that the replacement of $\hat{H}_{\bar{k}} = \hat{H}^1 \cup \hat{H}^3$ with $cl(\hat{H}_{\bar{k}}) = \hat{H}^1 \cup \hat{H}^2 \cup \hat{H}^3$ does not affect the allocation from the $\bar{k}$-th step TTC. If no $h < \hat{H}^1$ and $h^1 \in \hat{H}^1$ (no $h > \hat{H}^3$ and $h^3 \in \hat{H}^3$) such that h^1 is reachable from h (h^3 is reachable from h) in $\bar{k}-1$ steps, Lemma 9 implies what we want. If instead, there are $h < \hat{H}^1$ and $h^1 \in \hat{H}^1$ such that h^1 is reachable from h (symmetrically there are $h > \hat{H}^3$ and $h^3 \in \hat{H}^3$ such that h^3 is reachable from h) in $\bar{k}-1$ steps, Lemma 10 implies that every $h^2 \in \hat{H}^2$ is also reachable from h in $\bar{k}-1$ steps, which in turn implies what we want.

C Proof of Theorem3

Let $P_0 \in \mathcal{P} \backslash \mathcal{SP}_<$ be an arbitrary non-single-peaked preference. Let $h_k = \tau(P_0)$, then there are h_i, h_j such that $h_i < h_j < h_k$ and $h_i \ P_0 \ h_j$. (The symmetric case where $h_k < h_j < h_i$ can be handled similarly.) Consider the STTC-N rule $\varphi^S : \{\mathcal{SP}_< \cup \{P_0\}\}^I \to M$ defined by

$$S = (cl(\{h_i, h_j\}), cl(\{h_j, h_k\}), H) \,.$$

Let $P \in \{\mathcal{SP}_< \cup \{P_0\}\}^I$ be a profile of preferences such that $P_i = P_j = P_0$, $P_k \in \mathcal{SP}_<$ such that $h_j \ P_k \ h_k \ P_k \ h$ for all $h \neq h_j, h_k$, and for all $l \neq i, j, k$ let $P_l \in \mathcal{SP}_<$ such that $e(l) = h_l$. The figure below illustrates the preferences.



It is evident that $\varphi_i^S(P) = h_i$. Let agent i misreport another preference $P_i' \in \mathcal{SP}_<$ such that $\tau(P_i') = h_k$. Then $\varphi_i^S(P_i', P_{-i}) = h_k$ which is better than h_i under agent P_i.

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