On the Decomposability of Fractional Allocations*

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Abstract

A common practice in dealing with the allocation of indivisible objects is to treat them as infinitely divisible and specify a fractional allocation, which is then implemented as a lottery on integer allocations that are feasible. The question we study is whether an arbitrary fractional allocation can be decomposed as a lottery on an arbitrary set of feasible integer allocations. The main result is a characterization of decomposable fractional allocations, that is obtained by transforming the decomposability problem into a maximum flow problem. We also provide a separate necessary condition for decomposability.

Keywords: Indivisibility; Fractional allocation; Decomposability; Maximum flow

JEL Classification: C78, D82

1 Introduction

This paper examines the allocation of indivisible objects in a set up where money transfers are prohibited. An inevitable issue this literature has to confront is the lack of *ex-post* fairness. To deal with this issue, a common practice adopted by economists is to specify a fractional allocation, which treats objects as if they are infinitely divisible, and then implement it as a lottery on feasible integer allocations. This method was introduced by Hylland and Zeckhauser (1979) for the problem of assigning individuals to positions and it was subsequently adopted in mechanism design problems such as scheduling, house allocation, school choice, and course allocation (Crès and Moulin, 2001; Bogomolnaia and Moulin, 2001; Kesten and Ünver, 2015; Budish, 2011).

A central question underlying this methodology is hence whether or not the specified fractional allocation is decomposable, i.e., whether it can be expressed as a lottery on the set of feasible integer allocations. The answer to this question apparently depends on the constraints that identify the set of feasible integer allocations. The celebrated Birkhoff-von Neumann theorem (Birkhoff, 1946; Von Neumann, 1953) states, for the cases where the allocations are two-dimensional square matrices, that if the feasibility constraints are such that each row and each

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column sum to one, a fractional allocation is decomposable if and only if it satisfies the same constraints. Recently, Budish et al. (2013) study two-dimensional allocations, where every feasibility constraint requires that the sum of a particular subset of elements be within a preset range. They show that if the constraints that identify the feasible integer allocations confirm to a structure called "bihierarchy", a fractional allocation is decomposable if and only if it satisfies the same constraints.[1] Moreover, they show that if the feasibility constraints violate the bihierarchy structure but satisfy a richness condition, a fractional allocation satisfying these constraints may be indecomposable.

Our paper complements this literature and studies the decomposability issue of fractional allocations. However, we do not seek to characterize constraints on feasible integer allocations that precipitate the property that a fractional allocation is decomposable if and only if it satisfies the *same* constraints. Rather, we postulate that a mechanism designer, faced with a particular problem, has employed some mechanism that treats objects as infinitely divisible, and has specified a fractional allocation. The question we study is whether this fractional allocation can be decomposed as a lottery on a given set of feasible integer allocations. It is worth noting that we impose no restriction on the constraints that identify these feasible integer allocations, nor do we focus on the fractional allocations that satisfy the same set of constraints. As long as the feasibility of integer allocations is clearly defined, our results apply.

The main result is a characterization of decomposable fractional allocations, which we obtain by transforming the original decomposability problem into a maximum flow problem, which is more transparent and simpler than the original problem. In particular, given a fractional allocation and the set of feasible integer allocations, we construct a maximum flow problem with vertex capacities, which is standard, except for a class of additional equality constraints. We then vary the lottery on the set of feasible integer allocations so as to maximize the maximum flow. It is shown that the given fractional allocation is decomposable if and only if the maximized maximum flow is equal to the "grand sum" (i.e., the summation of all elements) of the fractional allocation. Moreover, in addition to checking whether the given fractional allocation is decomposable or not, our method also yields a decomposition whenever the fractional allocation is decomposable.[2]

Our characterization of decomposability complements the results by Birkhoff (1946), Von Neumann (1953), and Budish et al. (2013), in the following three ways. First, we do not require the allocations to be two-dimensional, nor do we require that the constraints identifying feasible integer allocations take particular formats. Moreover, the constraints studied by the aforementioned papers are such that the summation of a subset of elements in the allocation is in a preset range. (For detailed discussion, please refer to Section 1.2.) However, there could be situations where the context-specific constraints cannot be modeled in this way. For example, consider the problem of allocating three objects (a, b, and c) among three agents (1, 2, and 3). Suppose,

[1] The constraints in the Birkhoff-von Neumann theorem satisfy the bihierarchy structure. A more detailed discussion can be found in the section on related literature.

[2] In the situations where the given fractional allocation is not decomposable, our method leads to a lottery that might be chosen as an approximation to the given fractional allocation. For applications, we provide Matlab files, which are available online and a manual is given in Appendix B. We also give a demonstration, in which we randomly generate a fractional allocation and then approximate it by the aforementioned method.

for a specific problem, it is required that agent 1 gets a whenever agent 2 gets b. Since we impose no restriction on constraint formats, our results apply. It is also worth mentioning that our results can be applied to verify the decomposability of plane-stochastic matrices, the higher dimensional analogues of bi-stochastic matrices, while the aforementioned results do not apply. Put otherwise, our results apply to a larger set of problems.

Second, in the model setting studied by the aforementioned papers, if the constraints satisfy the bihierarchy structure, we know that every fractional allocation satisfying the same constraints is decomposable. However, the problem of finding a decomposition persists: Our characterization gives a specific method (see the leading example). In view of the fact that linear programming, in particular the maximum flow problem, has been intensively studied for decades and several algorithms that solve such problems are available, we believe that our method is practically useful.[3]

Third, in the model setting studied by the aforementioned papers, if the constraints violate the bihierarchy structure, we know that not all fractional allocations satisfying these constraints are decomposable. The question of how to ascertain whether or not a specific fractional allocation is decomposable persists: Our characterization applies to answer this question (see the leading example). Moreover, we also present in this paper is a separate necessary condition for checking decomposability.

In the remainder of the introduction we present a leading example, followed by a review of the literature. Section 2 introduces preliminary notations and definitions. Section 3 presents main results and Section 4 contains final remarks. The appendix collects material which can be omitted without harming the understanding of the main text.

1.1 A Leading Example

We consider the case where three indivisible objects (a, b, and c) are to be assigned among three agents (1, 2, and 3). We depart from the Bogomolnaia and Moulin (2001) formulation by allowing the agents to receive bundles of the goods. Thus there are in total 8 bundles, including the empty bundle. Each agent has a strict preference on bundles of objects. Suppose that in order to achieve ordinal efficiency, the probabilistic serial mechanism is adopted to specify the fractional allocation.[4] Consider the two preference profiles P and P' specified below and the correspondingly specified fractional allocations M and M', where to save space, in both pref-

[3]The simplex algorithm is classical for solving linear programmings and the Ford-Fulkerson algorithm is classical for the maximum flow problems. More algorithms can be found in Ahuja et al. (1993).

[4]The probabilistic serial mechanism (PS) was introduced by Bogomolnaia and Moulin (2001) for the allocation problem where each agent is supposed to get exactly one object. This mechanism satisfies a desirable property called ordinal efficiency, which requires that, no matter what are agents' cardinal utility representing their preferences, the specified fractional allocation is Pareto efficient under the assumption that agents are expected-utility maximizers. In this case, the fractional allocations generated by the PS mechanism are bi-stochastic matrices. The PS mechanism was generalized by Budish et al. (2013) to the setting where the fractional allocations are structured as "bihierarchy" (See more details in Section 1.2). More recently, Chatterji and Liu (2020) generalized the PS mechanism for the setting where each agent receives a bundle (a combination of different types of objects), and hence the fractional allocations are not bihierarchies. Aziz and Brandl (2020) proposed a unified model to incorporate various settings where the PS mechanism applies.

erences and fractional allocations we depict only those bundles that are allocated with positive probability in the fractional allocations. For instance, agent 1's preference indicates that the bundle containing a and b is the best. The second best is the singleton bundle containing only a. In particular, the empty bundle $\emptyset$ is one containing no object.

The question is whether or not M and M' are decomposable.[5]

$$P = \begin{pmatrix} P_1: & ab \succ a \succ b \succ c \succ \emptyset \succ \cdots \\ P_2: & b \succ a \succ ab \succ \emptyset \succ c \succ \cdots \\ P_3: & a \succ b \succ ab \succ \emptyset \succ c \succ \cdots \end{pmatrix} \qquad M = \begin{pmatrix} & ab & a & b & c & \emptyset & \cdots \\ M_1: & 1/2 & 0 & 0 & 1/2 & 0 & 0 \\ M_2: & 0 & 0 & 1/2 & 1/4 & 1/4 & 0 \\ M_3: & 0 & 1/2 & 0 & 1/4 & 1/4 & 0 \end{pmatrix}$$

$$P' = \begin{pmatrix} P_1: & ab \succ a \succ b \succ c \succ \emptyset \succ \cdots \\ P_2': & b \succ a \succ ab \succ c \succ \emptyset \succ \cdots \\ P_3': & a \succ b \succ ab \succ c \succ \emptyset \succ \cdots \end{pmatrix} \qquad M' = \begin{pmatrix} & ab & a & b & c & \emptyset & \cdots \\ M_1': & 1/2 & 0 & 0 & 1/3 & 1/6 & 0 \\ M_2': & 0 & 0 & 1/2 & 1/3 & 1/6 & 0 \\ M_3': & 0 & 1/2 & 0 & 1/3 & 1/6 & 0 \end{pmatrix}$$

From the equality below, we see that M is decomposable. In particular, the feasibility constraints in this example are apparent: each agent gets one bundle and each object is allocated to one agent. It is evident that the integer allocations below are all feasible. The assigned probabilities can be identified easily by applying Theorem 1 (the details can be found in Example 6).

$$M = \frac{1}{4}\begin{pmatrix} D^1 & ab & c & \emptyset \\ 1: & 1 & 0 & 0 \\ 2: & 0 & 1 & 0 \\ 3: & 0 & 0 & 1 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} D^2 & ab & c & \emptyset \\ 1: & 1 & 0 & 0 \\ 2: & 0 & 0 & 1 \\ 3: & 0 & 1 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} D^3 & a & b & c \\ 1: & 0 & 0 & 1 \\ 2: & 0 & 1 & 0 \\ 3: & 1 & 0 & 0 \end{pmatrix}$$

It is worth noting that the structure of fractional allocations here is different from that of the bi-stochastic matrices. In particular, for a bi-stochastic matrix, each row sums to one and each column sums to one. Here, each object's feasibility constraint requires several columns sum to one. (For a formal description of the feasibility, please refer to Example 3.) This feature tells that the Birkhoff-von Neumann theorem does not apply and so it is not easy to determine whether or not M' is decomposable. Our main result, Theorem 1, applies and reveals that M' is not decomposable. (For details, please refer to Example 7.)

[5] We explain the probabilistic serial mechanism with P. Imagine that the objects are infinitely divisible and there is a timer. At $t = 0$, all agents start to "eat" their respectively favorite bundle, at the unit speed. In particular, agent 1 eats simultaneously a and b, agent 2 eats only b, and agent 3 eats only a. Hence, both of a and b are exhausted at time $t = 1/2$, after when agents start to eat their favorite bundle among the available ones. In particular, agent 1 eats c and the others choose to eat nothing (bundle $\emptyset$). Since all objects need to be assigned, agents 2 and 3 have to stop eating $\emptyset$ and start eating c at $t = 3/4$. Suppose otherwise, they eat $\emptyset$ until $t = 5/6 > 3/4$, then the remaining share of c is $1 - (5/6 - 1/2)$, where the deducted share was eaten by agent 1. Then to eat up c, three agents need time $[1 - (5/6 - 1/2)]/3 = 2/9$. However, if so, the total share of bundles eaten by each agent is $5/6 + 2/9 = 19/18 > 1$. When shares are implemented as probabilities, this means that each agent has more than one probability of getting one bundle, a contradiction to the requirement that each agent receives one bundle. Hence, the second step ends at time $t = 3/4$ and in the third step all agents eat the remaining share of c. Finally, the shares of bundle eaten by agents are interpreted as the probabilities agents get these bundles.

1.2 Related Literature

This paper is to our understanding closely related to two strands of literature. The first is on the decomposability of matrices satisfying certain constraints. An important result in this literature is the decomposability of all bi-stochastic matrices as convex combinations of permutation matrices. This result is typically attributed to Birkhoff (1946) and Von Neumann (1953) and called the Birkhoff-von Neumann theorem. The traditional proof is by the so-called Birkhoff algorithm, which iteratively removes a fraction of a permutation matrix from the given bi-stochastic matrix, where the resulting matrix has at least one more zero entry. Then since the matrix is finite, the iteration terminates after finitely many steps.[6]

More recently, Budish et al. (2013) generalize the Birkhoff-von Neumann theorem to a larger class of feasibility constraints. In particular, they study two dimensional real matrices M of size $|I| \times |O|$, where I denotes the set of agents and O the set of objects, with the constraints in the form of $\underline{q}_S \leqslant \sum_{s \in S} M(s) \leqslant \overline{q}_S$, where $S \subset I \times O$ is a subset of indices and $\underline{q}_S, \overline{q}_S$ are fixed real numbers. These subsets of indices are collected and called a constraint structure, denoted as $\mathcal{S}$. The authors then introduce a condition on $\mathcal{S}$ called "bihierarchy", which requires that $\mathcal{S}$ can be partitioned into two subsets $\mathcal{S}_1, \mathcal{S}_2$ so that, for an arbitrary pair S, S' from $\mathcal{S}_1$ (or both from $\mathcal{S}_2$), either they are disjoint ($S \cap S' = \emptyset$) or one contains the other ($S \subset S'$ or $S' \subset S$). They show that whenever the constraint structure is a bihierarchy, all fractional allocations are decomposable as lotteries on the integer allocations defined by the same bihierarchy. Loosely speaking, bihierarchy requires that $\mathcal{S}$ can be partitioned into one set of "row constraints" and one set of "column constraints." A row(column) constraint refers to a subset of rows(columns) or a subset of one row(column). It is evident that the set of matrices attaining the bihierarchy structure includes the bi-stochastic matrices as a proper subset. Unfortunately, such a decomposability result does not hold for the problems where the feasibility constraints do not follow bihierarchies. In particular, high-dimensional analogues of bi-stochastic matrices, called the plain-stochastic matrices, may be indecomposable. Put in mathematical terms, the extremal points of these matrices include not only integer allocations but also some fractional allocations. Mathematicians are hence interested in characterizing the set of extremal points.[7]

The central difference between the current paper and the studies in this strand is that the question the latter study is whether *all* fractional allocations satisfying a particular set of constraints are decomposable or not, given that the set of feasible integer allocations is identified by the same set of constraints. Our question is whether an arbitrary fractional allocation is decomposable or not, given an arbitrary set of feasible integer allocations. In particular, we impose no

[6]The existence of such a permutation matrix is guaranteed by the Hall's marriage theorem. Another constructive proof of the Birkhoff-von Neumann theorem is by Hylland and Zeckhauser (1979). This proof iteratively decomposes the given bi-stochastic matrix as a convex combination of two bi-stochastic matrices, both of which have at least one more entry in $\{0, 1\}$. The idea of trading cycles is used in identifying the binary convex combination. The iteration terminates in finitely many step when all the bi-stochastic matrices become permutation matrices. There are quite a few different proofs of the Birkhoff-von Neumann theorem, including but not limited to Dantzig (1951), Chvatal et al. (1983), Romanovsky (1994), Hoffman and Wielandt (2003), and Hurlbert (2008).

[7]Representing studies are Brualdi and Csima (1975b), Brualdi and Csima (1975a), Csima (1970), Jurkat and Ryser (1968), Marchi and Tarazaga (1979).

structure on feasibility constraints, nor a relationship between the fractional allocation and set of feasible integer allocations.

The second strand of literature related to our paper is on the approximation of fractional allocations, given that the given fractional allocation is not decomposable. There are, to our knowledge, two recent papers on this topic. Akbarpour and Nikzad (2020) suggest that if the constraint structure $\mathcal{S}$ can be partitioned into two parts $\mathcal{P}$ and $\mathcal{Q}$ such that $\mathcal{P}$ is a bihierarchy and $\mathcal{Q}$ satisfies a condition defined with respect to $\mathcal{P}$, then there is a lottery on integer matrices such that the constraints in $\mathcal{P}$ are met exactly and the constraints in $\mathcal{Q}$ are met arbitrarily closely. This result is useful when some constraints are treated as "soft". For example, the constraint that a school has at least 50% of its students live within walk zone can be treated as soft, since, if necessary, 48% is also acceptable. Nguyen et al. (2016) impose the restriction on a fractional allocation that disallows an agent to receive a bundle containing more than k objects and show that it can be expressed as a lottery on integer allocations which over-allocates each object by at most $k-1$ units. The approximation of indecomposable fractional allocations is not the focus of our paper. However, our characterization of decomposability suggests an approximation method, as remarked in Section 4.

2 Preliminaries

Let $\mathcal{I}$ denote an exogenously given finite set of indices. Such a set is usually a Cartesian product, but we do not require it be one. A **fractional allocation** is a mapping $M : \mathcal{I} \to [0,1]$ specifying at each index a real number in between zero and one. An integer allocation is a special case of fractional allocations where either one or zero is specified at each index. Formally, an **integer allocation** is a mapping $D : \mathcal{I} \to \{0,1\}$. For specific allocation problems, it may not be true that all integer allocations are feasible. We henceforth denote the set of **feasible** (integer) allocations as $\mathcal{D}$. We say a fractional allocation is decomposable through $\mathcal{D}$ if it can be expressed as a convex combination of the integer allocations in $\mathcal{D}$.[8]

Definition 1. *A fractional allocation $M : \mathcal{I} \to [0,1]$ is **decomposable** through a given set of feasible allocations $\mathcal{D}$ if there is a lottery $\beta \in \triangle(\mathcal{D})$ such that $M = \sum_{D\in\mathcal{D}} \beta(D) \cdot D$.*

The grand sum of an allocation refers to the summation of all elements in it and is denoted $|M| \equiv \sum_{u\in\mathcal{I}} M(u)$. Note that for an integer allocation, its grand sum $|D|$ is exactly the number of ones in it. We assume that the feasible allocations have the same grand sum and denote $n \equiv |D|$ for all $D \in \mathcal{D}$.[9] It is evident that $|M| = n$ is a necessary condition for a fractional allocation M to be decomposable through $\mathcal{D}$. We henceforth assume $|M| = n$.

The requirements that identify the feasible allocations are called feasibility constraints. For different allocation problems, these constraints may take various forms. We list below some

[8] Note that the only relation between the fractional allocation to be implemented and the set of feasible integer allocations is that their index set is the same, which is evidently a necessary condition for decomposability. In particular, we do not require the fractional allocation to satisfy the feasibility requirement that identifies $\mathcal{D}$.

[9] It is true that in some allocation problems, feasible allocations may specify different numbers of ones. However, by introducing "null" indices, it is easy to transform the given feasible allocations to ones where all of them have the same grand sum. Please also refer to Footnote 13 for a specific example.

interesting examples.

Example 1. *Bogomolnaia and Moulin (2001) consider the allocation of n objects among n agents where each agent receives exactly one object. In this problem, the index set contains n^2 elements, i.e., $\mathcal{I} = \{1, \cdots, n\}^2$. The feasibility requirement that identifies the set of feasible allocations contains two classes of constraints, called the row constraints and the column constraints. A row constraint is due to the requirement that each agent receives exactly one object and a column constraint is due to the requirement that each object be assigned to exactly one agent. Hence an integer allocation $D : \mathcal{I} \to \{0, 1\}$ is feasible if and only if it satisfies the following constraints.*

$$\begin{aligned} &\textit{Row constraints:} && \forall i = 1, \cdots, n: \ \sum_{j=1}^{n} D(i,j) = 1 \\ &\textit{Column constraints:} && \forall j = 1, \cdots, n: \ \sum_{i=1}^{n} D(i,j) = 1 \end{aligned}$$

The set of feasible allocations $\mathcal{D}$ identified is evidently the set of permutation matrices. It is evident that a decomposable fractional allocation must also satisfy the above constraints. The question is whether the converse is true. The celebrated Birkhoff-von Neumann theorem gives an affirmative answer and states that a fractional allocation is decomposable through $\mathcal{D}$ if and only if it is a bi-stochastic matrix, i.e., a fractional allocation satisfying the constraints above. ■

Example 2. *Budish et al. (2013) consider a multi-sided allocation problem (whereas the previous model considered by Bogomolnaia and Moulin (2001) is a two-sided one). For example, students may be assigned to different schools and after-school programs. Assume that there are three sides where each side has n objects and so the index set in this setting is $\mathcal{I} = \{1, \cdots, n\}^3$. Analogous to the two-sided setting, an integer allocation in the current setting is feasible if and only if it satisfies the following three classes of constraints.*

$$\begin{aligned} &\forall i = 1, \cdots, n: && \sum_{j=1}^{n} \sum_{k=1}^{n} D(i,j,k) = 1 \\ &\forall j = 1, \cdots, n: && \sum_{i=1}^{n} \sum_{k=1}^{n} D(i,j,k) = 1 \\ &\forall k = 1, \cdots, n: && \sum_{i=1}^{n} \sum_{j=1}^{n} D(i,j,k) = 1 \end{aligned}$$

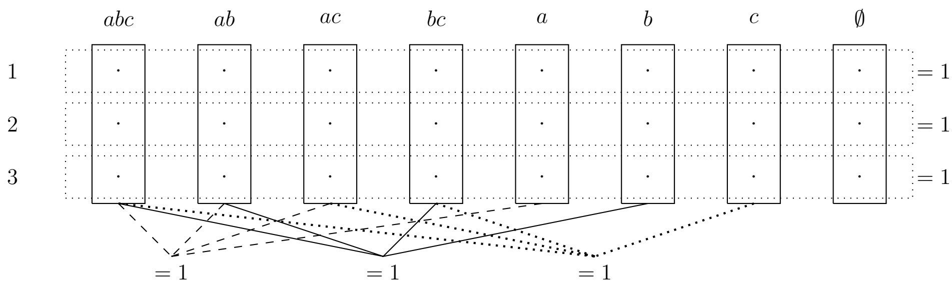
Higher-dimensional analogues of bi-stochastic matrices are called the plane-stochastic matrices, i.e., the matrices $M : \mathcal{I} \to [0, 1]$ satisfying the constraints above. These matrices have attracted the interest of mathematicians decades ago, see for example Csima (1970). It is evident that a decomposable fractional allocation must be a plane-stochastic matrix. It is also well-known that the converse is not true, i.e., not every plane-stochastic matrix is decomposable via integer allocations satisfying the above constraints. However, the question of whether or not a given plane-stochastic matrix is decomposable has been left uninvestigated. This question can be answered by applying Theorem 1.

Example 3. *(Leading example continued) Chatterji and Liu (2020) consider a two-sided allocation problem where each agent may receive a bundle rather than exactly one object. Suppose that three types of objects, a, b, and c, are to be allocated among three agents. For simplicity, assume that we have only one item of each type. Then a bundle is one in the set $\mathcal{B} \equiv \{abc, ab, ac, bc, a, b, c, \emptyset\}$. Then the index set is $\mathcal{I} = \{1,2,3\} \times \mathcal{B}$. An integer allocation is feasible if and only if it satisfies the following constraints.*

$$\text{Each agent receives one bundle} \qquad \forall i \in \{1,2,3\} : \sum_{B \in \mathcal{B}} D(i,B) = 1$$

$$\text{Each object is assigned to one agent} \quad \forall x \in \{a,b,c\} : \sum_{i=1}^{3} \sum_{\{B \in \mathcal{B} : x \in B\}} D(i,B) = 1$$

The above is a formal presentation of the model setting discussed in the leading example. The feasibility requirement in this setting exhibits a combinatorial pattern, which is the key feature that distinguishes it from the setting of Bogomolnaia and Moulin (2001). In particular, each constraint here requires certain columns together, rather than a single column, sum to a fixed number. For a better illustration, we visualize this point as follows.



It is evident that a decomposable fractional allocation in this setting must satisfy the constraints listed above.[10] *The converse is not true, as already noticed by Chatterji and Liu (2020). However, the question of whether or not a given fractional allocation satisfying the aforementioned constraints is decomposable or not has been left uninvestigated.* [11] ■

Example 4. *All the feasibility constraints in the previous examples take the form of equalities. However, constraints in some allocation problems may appear as inequalities, as is the case for school choice problems.*[12] *In particular, each school has a certain number of seats available,*

[10] It is true for any linear constraint that, if all feasible integer allocations satisfy it, then a decomposable fractional allocation must also satisfy it. A linear constraint can be written as an inequality $A \cdot D \leqslant b$, where $A : \mathcal{I} \to \mathbb{R}$ is a real matrix of the same size as the matrix D and $b \in \mathbb{R}$ is a real scaler. The dot product denotes the inner product, i.e., the summation of the products of the paired elements in A and D at the same index. It is evident that the constraints in Example 1 to 3 are all linear constraints. (An equality constraint is equivalent to the combination of two inequality constraints.) Hence a necessary condition for a fractional allocation to be decomposable is that it satisfies the linear constraints that identify the feasible allocations.

[11] A fractional allocation satisfying such a feasibility requirement can be written as a high-dimensional matrix attaining a similar pattern as a plane-stochastic matrix, as shown in Appendix A.

[12] Another allocation problem where inequality constraints arise is course allocation, as the capacity of a course is usually not a fixed quantity but a range between a upper bound and a lower bound of students. Please refer to Budish and Cantillon (2012) for detailed discussions.

which can be disposed of with no cost. To be specific, suppose 6 *students are to be assigned to* 3 *schools, denoted* S_1, S_2, S_3*, where* S_1 *can admit at most four students,* S_2 *two, and* S_3 *one. The index set is then* $\mathcal{I} = \{1, \cdots, 6\} \times \{S_1, S_2, S_3\}$*. Hence, a feasible integer allocation must satisfy the following constraints.*[13]

Each student is admitted to one school:	$\forall i = 1, \cdots, 6 : \sum_{S=S_1,S_2,S_3} D(i, S) = 1$
Each school admits no more students than its capacity:	$\sum_{i=1}^{6} D(i, S_1) \leqslant 4 \quad \sum_{i=1}^{6} D(i, S_2) \leqslant 2 \quad \sum_{i=1}^{6} D(i, S_3) \leqslant 1$

In addition to these constraints, a feasible integer allocation is usually required to meet certain policies. Consider for example that priorities are specified according to whether a student lives in walkable zone and whether she has siblings already in a particular school. Suppose the priorities are public and verifiable information and are as described below. In particular, student 3 *has the highest priority in* S_1 *as she lives in the walkable zone of* S_1 *and has siblings in it. Students* 1 *and* 2 *have the second highest priority in* S_1 *as they live in the walkable zone, but don't have siblings in this school. The remaining priorities can be read similarly.*

	S_1	S_2	S_3
Walk zone and sibling	3	–	–
Walk zone but no sibling	$1, 2$	5	–
Sibling but not walk zone	$4, 5$	–	–

The policy alone can not judge whether an integer allocation is feasible or not. To do so, we need information of students' preferences, which are assumed as in the left panel below.

$1 : \quad S_1 \succ_1 S_2 \succ_1 S_3$

$2 : \quad S_2 \succ_2 S_3 \succ_2 S_1$

$3 : \quad S_1 \succ_3 S_2 \succ_3 S_3$

$4 : \quad S_1 \succ_4 S_2 \succ_4 S_3$

$5 : \quad S_2 \succ_5 S_1 \succ_5 S_3$

$6 : \quad S_1 \succ_6 S_2 \succ_6 S_3$

$$\begin{array}{c|ccc|c}
 & S_1 & S_2 & S_3 & \\
\hline
1 & 1 & 0 & 0 & = 1 \\
2 & & & & = 1 \\
3 & 1 & 0 & 0 & = 1 \\
4 & 1 & 0 & 0 & = 1 \\
5 & 0 & 1 & 0 & = 1 \\
6 & & & & = 1 \\
\hline
 & \leqslant 4 & \leqslant 2 & \leqslant 1 &
\end{array}$$

In particular, since 3*'s favorite is* S_1 *and at this school she has the highest priority,* $D(3, S_1) = 1$ *must be true for a feasible integer allocation. Similarly, we have* $D(1, S_1) = D(4, S_1) =$

[13] In practice, students may have outside option, which tends to imply that the feasibility constraints for students are also inequalities. However, by assuming the existence of a null school representing outside options, the equality constraints here are without loss of generality.

$D(5, S_2) = 1$. *It is evident that an integer allocation is feasible if and only if it satisfies the constraints aforementioned and specifies ones at these indices, as illustrated in the right panel above. The literature has focused on the random deferred acceptance algorithm for this problem as it not only satisfies the constraints mentioned above but also delivers an ex post stable matching (Kesten and Ünver, 2015). However, there remains the problem of deciding whether an arbitrary fractional allocation in this set up is decomposable: Our method applies.*

3 Results

This section presents our main results: a characterization of decomposable fractional allocations and a separate necessary condition. Throughout this section, we fix an arbitrary index set $\mathcal{I}$ and the set of feasible integer allocations $\mathcal{D}$. We begin with a brief introduction to the maximum flow problem and the classical results which we will invoke later.

3.1 A Preparation on the Maximum Flow Problem

A directed graph is a collection of two sets (V, E), where V is a finite set of vertices and $E \subset V \times V$ is the set of edges. An edge $e \in E$ is usually denoted uv rather than (u, v). We call $e = uv$ an edge from u to v and denote its start $e_1 = u$ and end $e_2 = v$.

A maximum flow problem is described as a tuple $G = (V, E, s, t, c)$ where (i) (V, E) is a directed graph, (ii) $s, t \in V$ are vertices indicating the source and sink, and (iii) $c : V\backslash\{s, t\} \to \mathbb{R}_+$ indicates the capacities of intermediate vertices. A flow is a vector $x \in \mathbb{R}_+^E$ which specifies a volume on each edge. The maximum flow problem is a linear programming, formally presented below.[14]

$$
\begin{aligned}
\max_{x \in \mathbb{R}_+^E} \quad & \sum_{e_2 = t} x_e \\
\text{s.t.} \quad & \sum_{e_2 = v} x_e = \sum_{e_1 = v} x_e, \forall v \in V\backslash\{s, t\} \quad && \text{(flow conservation)} \\
& \sum_{e_2 = v} x_e \leqslant c(v), \forall v \in V\backslash\{s, t\} \quad && \text{(capacity constraints)}
\end{aligned}
$$

A feasible flow is one satisfying two sets of constraints. First, the sum of volumes entering a vertex must be equal to the sum of volumes exiting that vertex, except for the source and sink. Second, the sum of volumes entering (and hence exiting) a vertex other than the source and sink must be no higher than the capacity of that vertex.

There are quite a few algorithms available for finding maximum flows, among which is the celebrated Ford-Fulkerson method (Ford and Fulkerson, 1956). Another useful result is the min-cut max-flow theorem. In particular, a subset of vertices $U \subset V\backslash\{s, t\}$ is called a cut if there is no path from the source to sink in the residual graph, which results when the

[14] A standard maximum flow problem defines capacities on edges rather than vertices. However, it is easy to transform the problem here to a standard one with edge capacities. In particular, we split each vertex $v \in V\backslash\{s, t\}$ into two vertices v_{in} and v_{out}, add one edge $v_{in}v_{out}$ with capacity $c(v)$, and then convert every edge uv to an edge uv_{in} with capacity $+\infty$ and every edge vu to an edge $v_{out}u$ with capacity $+\infty$. Detailed discussion can be found in Ahuja et al. (1993).

vertices in U and related edges are removed.[15] The capacity of a cut is simply the summation of capacities attached to these removed vertices. The min-cut max-flow theorem then states that the maximum flow is equal to the minimum capacity among all cuts. Furthermore, for each vertex in a minimum cut, the corresponding capacity constraint is binding.

3.2 Characterizing Decomposability

Let $M : \mathcal{I} \to [0, 1]$ be an arbitrary fractional allocation and $\beta \in \triangle(\mathcal{D})$ an arbitrary lottery. We define a maximum flow problem, where the capacities depend on both M and β.

Let $G_{M,\beta} \equiv (V, E, s, t, c_{M,\beta})$ be as follows.

- $V = \mathcal{I} \cup \mathcal{D} \cup \{s, t\}$;
- $E = E_1 \cup E_2 \cup E_3$, where
  - $E_1 = \{su|u \in \mathcal{I}\}$ includes an edge from the source to each vertex in $\mathcal{I}$,
  - $E_2 = \{uD|u \in \mathcal{I}, D \in \mathcal{D}$ and $D(u) = 1\}$ includes an edge from an index $u \in \mathcal{I}$ to an integer allocation $D \in \mathcal{D}$ if and only if D specifies 1 at this index (so there are exactly n edges leading to each D), and
  - $E_3 = \{Dt|D \in \mathcal{D}\}$ includes an edge from each vertex in $\mathcal{D}$ to the sink;
- $c_{M,\beta} : V\backslash\{s, t\} \to \mathbb{R}_+$ is such that $c_{M,\beta}(u) = M(u)$ for all $u \in \mathcal{I}$ and $c_{M,\beta}(D) = n \cdot \beta_D$ for all $D \in \mathcal{D}$.[16]

Given the description of $G_{M,\beta} \equiv (V, E, s, t, c_{M,\beta})$, the corresponding maximum flow problem is given below, and its value is denoted $\Gamma_{M,\beta}$.

$$\begin{aligned} \Gamma_{M,\beta} \equiv \max_{x \in \mathbb{R}^E_+} \quad & \sum_{e_2=t} x_e \\ \text{s.t.} \quad & \sum_{e_2=v} x_e = \sum_{e_1=v} x_e, \forall v \in V\backslash\{s, t\} \quad \text{(flow conservation)} \\ & \sum_{e_2=v} x_e \leqslant c_{M,\beta}(v), \forall v \in V\backslash\{s, t\} \quad \text{(capacity constraints)} \end{aligned} \tag{1}$$

Example 5. *(Leading example continued) Consider the allocation problem in Example 3. Given the constraints described there, it is evident that there are in total 27 feasible allocations. Let M be an arbitrary fractional allocation and β an arbitrary lottery on feasible allocations. We specify a maximum flow problem $G_{M,\beta} = (V, E, s, t, c_{M,\beta})$, which is illustrated in Figure 1. For simplicity, we draw only a part of the graph. We denote a feasible allocation by indicating the indices at which an one is specified. The numbers above the vertices denote capacities.* ■

We now consider an alternative problem, which is the same as the maximum flow problem (1) except that an additional set of equality constraints further shrinks the set of feasible flows. These constraints require that the same flow volume be assigned to the edges that lead to the

[15]The residual graph consists of the vertex set $V\backslash U$ and the edge set $\{e \in E : e \in (V\backslash U) \times (V\backslash U)\}$.
[16]Recall that n denotes the common grand sum of feasible allocations.

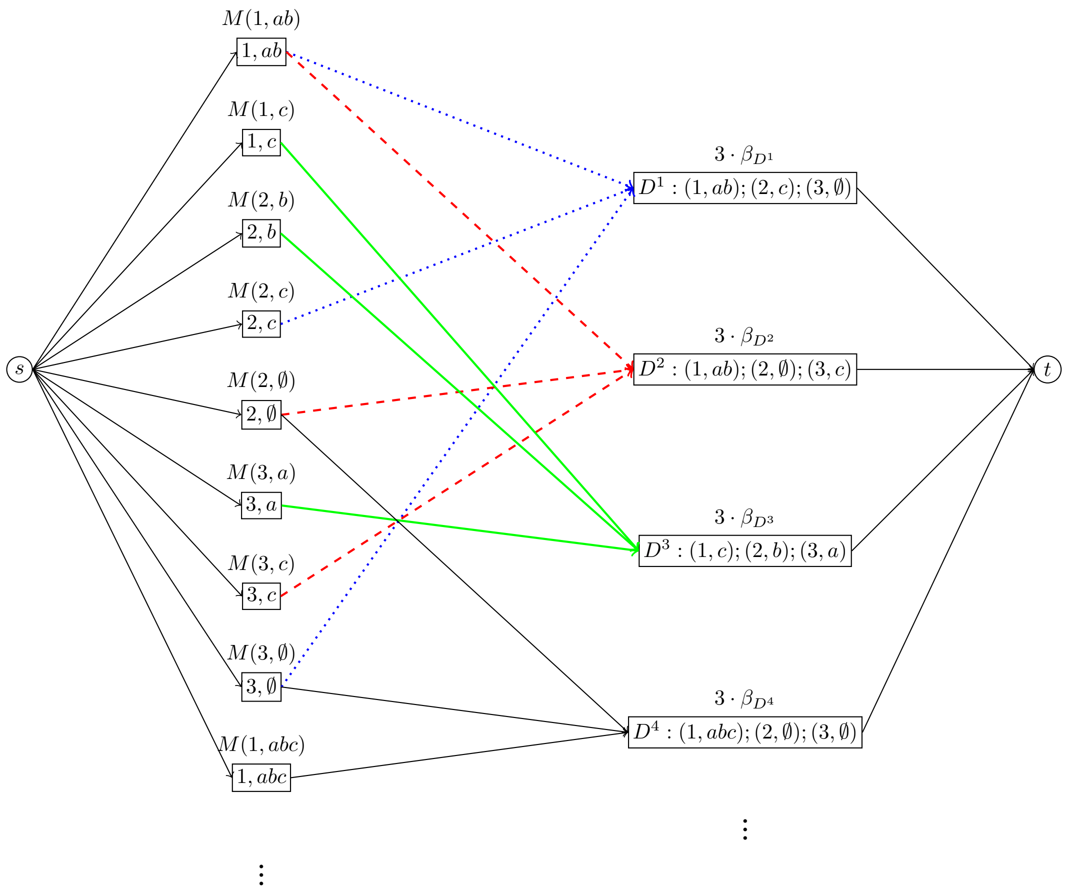


Figure 1: A Maximum Flow Problem

same feasible allocation. It turns out that these constraints are related to the decomposability of given fractional allocation. We hence call them decomposability constraints and denote the value of this new problem $\Psi_{M,\beta}$.

$$
\begin{aligned}
\Psi_{M,\beta} \equiv \max_{x \in \mathbb{R}^E_+} \quad & \sum_{e_2 = t} x_e \\
\text{s.t.} \quad & \sum_{e_2 = v} x_e = \sum_{e_1 = v} x_e, \forall v \in V \backslash \{s, t\} && \text{(flow conservation)} \\
& \sum_{e_2 = v} x_e \leqslant c_{M,\beta}(v), \forall v \in V \backslash \{s, t\} && \text{(capacity constraints)} \\
& x_e = x_{e'}, \forall e, e' \in E_2 \text{ s.t. } e_2 = e'_2 && \text{(decomposability constraints)}
\end{aligned}
\tag{2}
$$

The respective domains of the maximization problems in (1) and (2) are evidently both closed and bounded. Then the fact that the common objective function is continuous implies the existence of solutions to both problems so that $\Gamma_{M,\beta}$ and $\Psi_{M,\beta}$ are well-defined and $\Psi_{M,\beta} \leqslant \Gamma_{M,\beta}$. We require for our purpose that the value of problem (2) satisfy a particular property. Specifically, the theorem below states that a fractional allocation is decomposable if and only if there is a β such that the value of the above problem is exactly n, the common grand sum of feasible allocations.

Theorem 1. *A fractional allocation M is decomposable iff $n = \max_{\beta \in \triangle(\mathcal{D})} \Psi_{M,\beta}$.*

Proof. **Sufficiency:** To show the sufficiency, we state the following lemma, which characterizes the flow that solves (2) when $n = \max_{\beta \in \triangle(\mathcal{D})} \Psi_{M,\beta}$. This lemma is also useful when applying the theorem.

Lemma 1. *Let $\beta^* \in \triangle(\mathcal{D})$ be such that $\Psi_{M,\beta^*} = n$. Let in addition, $x^* \in \mathbb{R}_+^E$ be a flow that delivers Ψ_{M,β^*}. Then*

1. *$\forall su \in E_1$, $x^*_{su} = M(u)$;*
2. *$\forall uD \in E_2$, $x^*_{uD} = \beta^*_D$;*
3. *$\forall Dt \in E_3$, $x^*_{Dt} = n \cdot \beta^*_D$.*

Proof of Lemma 1: We prove first that all capacity constraints in (2) hold with equalities. It suffices to show that the capacity constraints in (1) hold with equalities. To see this, note that $V_2 = \mathcal{D}$ is a cut in G_{M,β^*}, whose capacity is n. Since (1) is a standard maximum flow problem, the min-cut max-flow theorem applies and implies that $\Gamma_{M,\beta^*} \leqslant n$, which together with $\Gamma_{M,\beta^*} \geqslant \Psi_{M,\beta^*} = n$, give $\Gamma_{M,\beta^*} = n$. Hence, the min-cut max-flow theorem implies that V_2 is indeed a minimum cut. Then the capacity constraint for each vertex $D \in \mathcal{D}$ holds with equality. Similarly, V_1 is also a minimum cut and hence the capacity constraint for each vertex $u \in V_1$ also holds with equality. Given that all capacity constraints hold with equalities, the flow conservation constraints imply 1, 3, and that $\forall D \in V_2$, $\sum_{\{e \in E_2 : e_2 = D\}} x^*_e = n \cdot \beta^*_D$. Then, since there are exactly n edges leading to each D, the decomposability constraints imply 2, which completes the proof of the lemma.

Given Lemma 1, $\forall u \in V_1$, we have the following equality, which completes the proof of sufficiency.

$$
\begin{aligned}
M(u) &= \sum_{\{e \in E_2 : e_1 = u\}} x^*_e && \text{by flow reservation} \\
&= \sum_{\{e \in E_2 : e_1 = u, e_2 = D, D(u) = 1\}} x^*_e && \text{by construction of graph} \\
&= \sum_{\{D \in \mathcal{D} : D(u) = 1\}} \beta^*_D && \text{by Lemma 1}
\end{aligned}
$$

Necessity: Let $\beta^* \in \triangle(\mathcal{D})$ be a decomposition of M, i.e., $M = \sum_{D \in \mathcal{D}} \beta^*_D \cdot D$. We construct a flow x^*, as follows.

$\forall\ su \in E_1$, let $x^*_{su} = M(u)$;
$\forall\ uD \in E_2$, let $x^*_{uD} = \beta^*_D$;
$\forall\ Dt \in E_3$, let $x^*_{Dt} = n \cdot .\beta^*_D$.

It can be verified that this flow satisfies all constraints in (2) and its value is exactly n. In addition, V_2 is a cut with capacity n, which means the value of (2) is at most n. Hence we have $n = \max_{\beta \in \triangle(\mathcal{D})} \Psi_{M,\beta}$. ∎

Remark 1. *As the maximum flow problem is essentially a linear programming, one may also prove the theorem directly, without invoking the max-flow min-cut theorem. We choose to establish the equivalence with this well-known result as it is convenient and serves as an application to the problem of decomposability.*

3.3 Discussion

We argue here that our transformation of the decomposability problem is an improvement over the original linear programming that defines the problem. By definition, the decomposability problem is to solve a system of $|\mathcal{I}|$ equations and $|\mathcal{D}|$ unknowns. Take for instance the leading example 1.1. To judge whether M or M' is decomposable, one needs to solve a system consisting of 24 equations, i.e. for each $(i, B) \in \{1, 2, 3\} \times \mathcal{B}$ an equation below.

$$M(i, B) - \sum_{D(i,B)=1} \beta_D = 0$$

Theorem 1 transforms the original definition to a maximum flow problem, which in our view is much more transparent. This transformation makes it possible to solve specific problems by a simple investigation on the network structure. Example 6 and 7 illustrate this point. Furthermore, as Example 6 below shows, our method also directly yields a decomposition whenever the fractional allocation is decomposable.

Example 6. *(Leading example continued) Consider the allocation problem in Example 3 and the maximum follow illustrated by Figure 1. From the equality in the leading example, we know that M is decomposable. We now apply Theorem 1 and Lemma 1 so as to identify the lottery presented there. Suppose β^* is a decomposition of M. By Lemma 1, the volumes assigned to the edges $s \to (1, abc)$ and $(1, abc) \to D^4$ are respectively 0 and $\beta^*_{D^4}$. Then the flow reservation condition implies $\beta^*_{D^4} = 0$. Similar argument applies and implies that $\beta^*_D = 0$ for all integer allocations except D^1, D^2, and D^3. Hence the flow maximization problem can be simplified as the one illustrated in Figure 2.*

We can then verify the decomposability of M by solving the flow maximization problem. To see this, note first that Lemma 1 implies that the volume assigned to edge $(1, c) \to D^3$ is $\beta^*_{D^3}$ and the volume assigned to $s \to (1, c)$ is $1/2$. Hence the flow conservation of $(1, c)$ implies $\beta^*_{D^3} = 1/2$. Similar arguments apply to the edges $(2, c) \to D^1$ and $(3, c) \to D^2$. Hence we have $\beta^*_{D^1} = \beta^*_{D^2} = 1/4$. ■

Example 7. *(Leading example continued) We now apply Theorem 1 and Lemma 1 to show that the fractional allocation M' in the leading example is indecomposable. Suppose not and β^* is a decomposition. First, by similar arguments in Example 6, $\beta^*_D = 0$ for all integer allocation D except D^1, D^2, and D^3. We have hence the flow maximization problem illustrated in Figure 3.*

By Theorem 1, to verify that M' is indecomposable, it suffices to check that the maximum flow of the problem in Figure 3 is less than 3 for any $\beta^*_{D^1}$, $\beta^*_{D^2}$, and $\beta^*_{D^3}$. To see this, note that the collection of vertices (i, B)s is a cut, whose capacity is $2\frac{5}{6}$. Hence the min-cut max-flow theorem implies the indecomposability of M'. ■

Besides transparency, the second advantage of our transformation, relative to solving the equation system as originally defined, is that it suggests a separate necessary condition for decomposability. We present it in subsection 3.4.

The last advantage of our transformation is that, when a fractional allocation is found indecomposable, a possibly good approximation of the decomposition is suggested by the maximum flow. The original problem of solving the equation system gives no information other than the

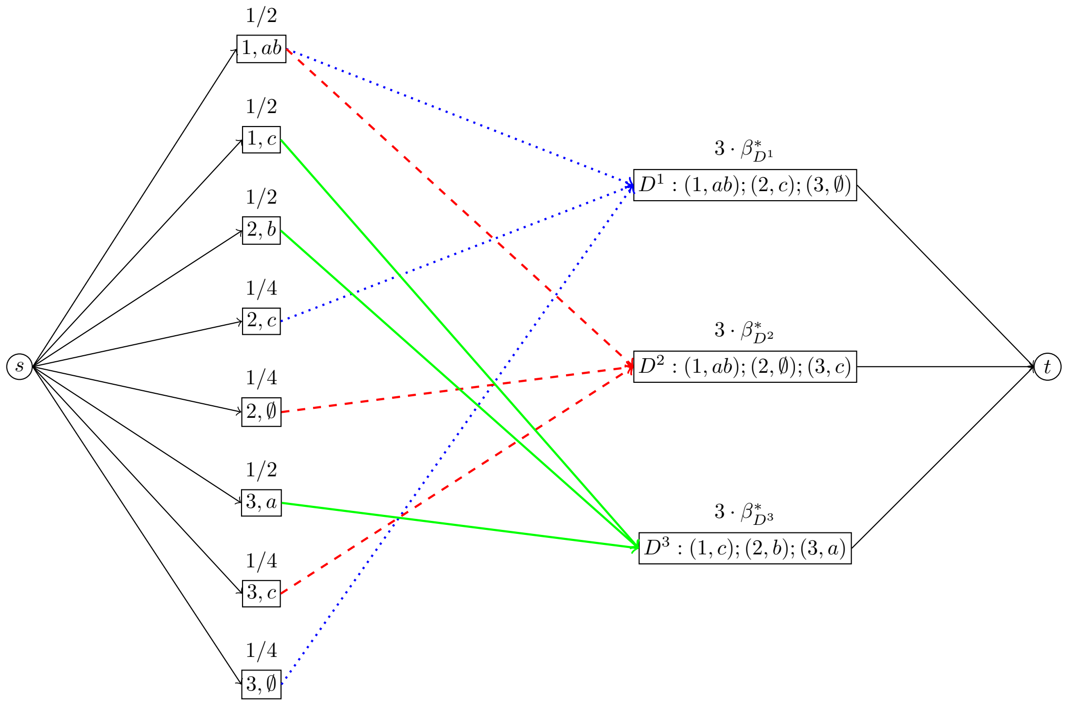


Figure 2: Checking Decomposability of M

conclusion of indecomposability. More concretely, indecomposability implies that the maximum flow is less than the grand sum of the fractional allocation. This maximum flow could nevertheless give an approximation of the decomposition. Since the approximation issue is beyond the scope of the current paper, we do not investigate in depth whether and in what sense this approximation is good. In stead, we provide a short discussion on this as a final remark (Section 4).

3.4 A Separate Necessary Condition

To induce the necessary condition, we reexamine the indecomposability of M'. The reader can refer to Figure 3 in Example 7. Suppose β^* is a decomposition of M', we derive a contradiction below. First, the capacity of $(3, \emptyset)$ requires that the volume assigned to the edge $(3, \emptyset) \to D^1$ is at most $M'(3, \emptyset) = 1/6$. Lemma 1 them implies that $\beta^*_{D^1} \leqslant 1/6$. Similar argument implies $\beta^*_{D^2} \leqslant M'(2, \emptyset) = 1/6$. Then the decomposability constraints of D^1 and D^2 imply respectively that the volumes assigned to edges $(1, ab) \to D^1$ and $(1, ab) \to D^2$ are both smaller than or equal to $1/6$. Next, note that $(1, ab)$ is connected to only D^1 and D^2. However the capacity of $(1, ab)$ is $1/2$, higher than $1/6 + 1/6 = 1/3$. One can then see that the flow specified in Lemma 1 does not fully utilize the capacity of $(1, ab)$ and hence has a value smaller than 3. Theorem 1 then implies the indecomposability of M'.

From the arguments above, one may conclude that M' is indecomposable because of the existence of an element in M', which is, loosely speaking, “too large”. In particular, $M'(1, ab) = 1/2$ is too large to be implemented through feasible integer allocations, given the other elements,

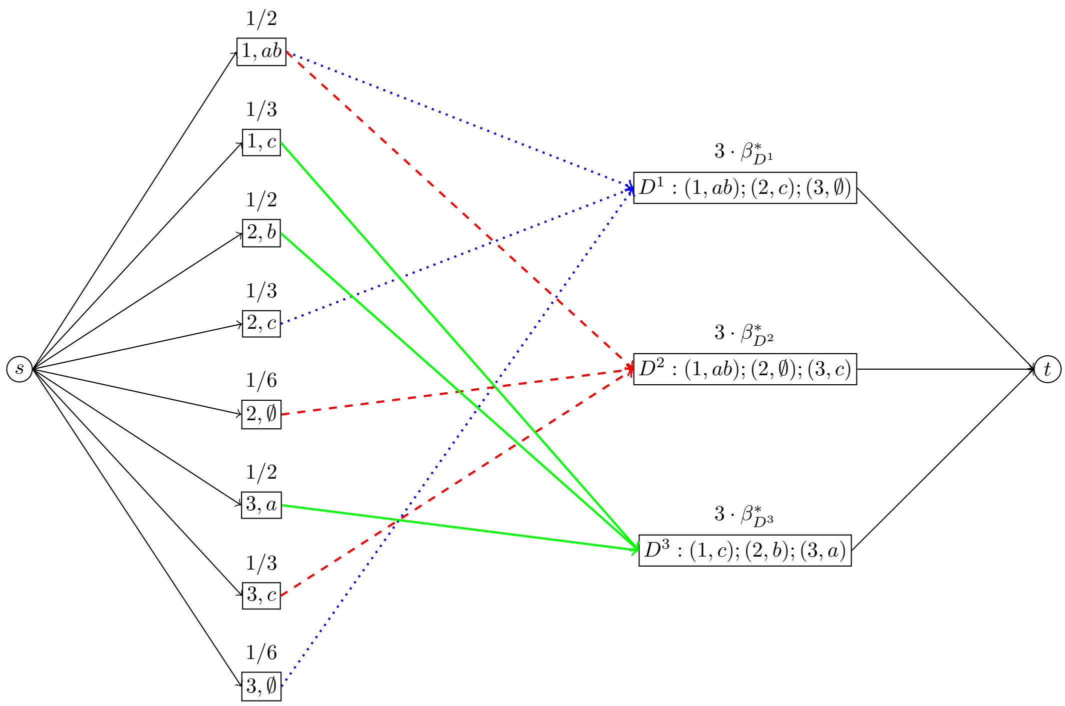


Figure 3: Checking Decomposability of M'

in particular $M'(2, \emptyset) = M'(3, \emptyset) = 1/6$. Hence a necessary condition for decomposability is the non-existence of such "too large" elements, which is formally presented below.

Proposition 1. *If M is decomposable through $\mathcal{D}$, then, $\forall u \in \mathcal{I}$*

$$M(u) \leqslant \sum_{\{D \in \mathcal{D}: D(u)=1\}} \min\{M(v) : D(v) = 1\}.$$

Proof. Let $M = \sum_{D \in \mathcal{D}} \beta_D \cdot D$. Fix an arbitrary index $u \in \mathcal{I}$, we have $M(u) = \sum_{\{D \in \mathcal{D}: D(u)=1\}} \beta_D$. Hence, it suffice to show $\beta_D \leqslant \min\{M(v) : D(v) = 1\}$ for all D such that $D(u) = 1$. Suppose not, and let $\beta_D > M(v)$ for some $v \in \mathcal{I}$ and D such that $D(u) = 1$. Then we have

$$\beta_D > M(v) = \sum_{\{D' \in \mathcal{D}: D'(v)=1\}} \beta_{D'} \geqslant \beta_D,$$

where the last inequality comes from $D \in \{D \in \mathcal{D} : D(u) = 1\} \cap \{D' \in \mathcal{D} : D'(v) = 1\}$. ■

This necessary condition could be used as a first step in checking for decomposability. For specific examples, it might be easier to identify some "too large" elements, as compared to constructing a maximum flow problem and then solving it. The example below proves the indecomposability of M' by applying the proposition. However, it should be noted that the condition in the above proposition is only a necessary condition for decomposability.

Example 8. *(Leading example continued) The following inequality indicates the indecomposability of M' in the leading example.*

$$\begin{aligned} M'(1,ab) = 1/2 > & \sum_{\{D\in\mathcal{D}:D(1,ab)=1\}} \min\{M'(v) : D(v) = 1\} \\ = & \min\{M'(v) : D^1(v) = 1\} + \min\{M'(v) : D^2(v) = 1\} \\ = & \min\{M'(1,ab), M'(2,c), M'(3,\emptyset)\} + \min\{M'(1,ab), M'(2,\emptyset), M'(3,c)\} \\ = & \min\{1/2, 1/3, 1/6\} + \min\{1/2, 1/3, 1/6\} = 1/6 + 1/6 = 1/3. \end{aligned}$$

■

4 Final Remarks

Our main result, i.e., Theorem 1, states that a given fractional allocation is decomposable through feasible allocations if and only if we can find a lottery β^* such that $\Psi_{M,\beta^*} = n$. We reiterate that (i) our theorem holds independently of the feasibility constraints, which means our result applies as long as the set of feasible allocations is clearly defined, and (ii) since linear programming, and in particular the maximum flow problem, have been intensively studied for decades, there are quite a few algorithms available for solving maximum flow problems, and this makes our results practically useful.

Finally, our method proposes an approximation when the given fractional allocation proves indecomposable. In such a situation, the message from Theorem 1 is that $\max_{\beta\in\Delta(\mathcal{D})} \Psi_{M,\beta} < n$. The structure here however suggests that the β^* that solves the maximization problem might nonetheless give a good approximation of M, i.e., $M^* \equiv \sum_{D\in\mathcal{D}} \beta^*_D \cdot D$ can serve as an approximate decomposition of M. We provide Matlab files for application and a manual in Appendix B. The natural question here is how well M^* approximates the originally given fractional allocation M. We next turn to this issue. Let the solution to problem (2) be β^*, the maximum flow be x^*, and its value be $n-\varepsilon$ where $\varepsilon > 0$. For an arbitrary index $u \in \mathcal{I}$, we are interested in the size of $|M^*(u) - M(u)|$. By the structure of the graph we constructed, $M^*(u) = \sum_{e:e_1=u,e_2\in\mathcal{D}} x^*_e$, whose value can not exceed the capacity of vertex u, i.e., $M(u)$. Put otherwise, what we need to know is the value of $M(u) - \sum_{e:e_1=u,e_2\in\mathcal{D}} x^*_e$, which measures the part of the capacity of u that is not utilized by x^*. This value depends apparently on β^*, which is endogenously determined and dependent on the structure of the graph, in particular, on the structure of edges in E_2. Note further that the edges in E_2 are basically dictated by the feasibility constraints that identifies the set of integer allocations. Hence, to investigate the precision of the approximation, a direction for study could focus on specific feasibility constraints that allow an explicit characterization of β. This line of enquiry is beyond the scope of the current paper, and we leave it for future work.

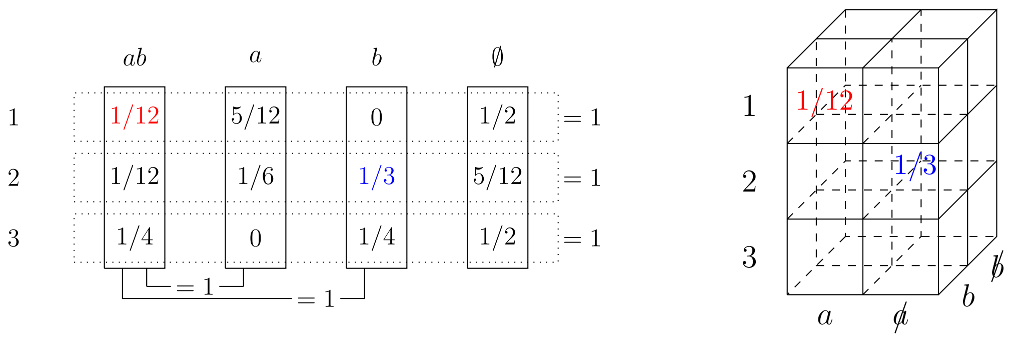
Appendix

A An equivalent modeling of the feasibility constraints in Example 3

For simplifying the exposition, we assume that two objects are to be allocated among three agents. Let $\mathcal{I} = \{1, 2, 3\} \times \{a, \not{a}\} \times \{b, \not{b}\}$. Consider the following constraints

$$\begin{aligned}
\forall i = 1, 2, 3: \quad & \sum_{j \in \{a, \not{a}\}} \sum_{k \in \{b, \not{b}\}} D(i, j, k) = 1 \\
\forall j = a, \not{a}: \quad & \sum_{i=1}^{3} \sum_{k \in \{b, \not{b}\}} D(i, j, k) = 1 \\
\forall k = b, \not{b}: \quad & \sum_{i=1}^{3} \sum_{j \in \{b, \not{b}\}} D(i, j, k) = 1
\end{aligned}$$

It is evident that the above constraints attain the same pattern as the ones in Example 2 for the plane-stochastic matrices. Note that the two-dimensional fractional allocation below is one that fits into the setting in Example 3. The three-dimensional matrix below is one that fits into the current setting. It is evident that these two presentations are equivalent. For example, the element at index $(1, ab)$ in the left panel corresponds to the element at $(1, a, b)$ in the right panel and $(2, b)$ corresponds to $(2, \not{a}, b)$.



B A manual of Matlab Files

The Matlab files we provide include the following:

1. Approx.m, which takes as inputs a fractional allocation M and the set of feasible integer allocations $\mathcal{D}$, and returns the approximation of M, denoted M^*, and the approximation error, measured as the average absolute value of the distance between the pairs of elements at the same index in M and M^*.[17]

[17] Another input of this function is "method", which can be either 'fmincon' or 'ga'. If 'fmincon' is given, local optimization in finding β^* will be performed and the initial point for searching local optima will be randomly generated. If 'ga' is given, global optimization will be performed by employing the generic algorithm, which will take longer time to find the solution.

2. objectivefun.m, which calculates the value of the problem in (2), taking as inputs the given fractional allocation, the set of feasible integer allocations, and a lottery on these integer allocations. This file is called by Approx.m.

3. creategraph.m, which constructs the graph illustrated in Figure 1, taking as inputs the given fractional allocation and the set of feasible integer allocations. This file is called by objectivefun.m.

4. plane-sto.m, which randomly generates a plane-stochastic matrix, given as inputs the number of dimensions and the number of indices in each dimension.[18]

5. plane-det.m, which identifies the set of all integer plane-stochastic matrices, taking as inputs the number of dimensions and the number of indices in each dimension.

Demo.m gives a demonstration where we randomly generate a plane-stochastic matrix M of size $3 \times 3 \times 3$ and then use Approx.m to identify an approximation of M, denoted M^*. In our simulations, it turns out that the approximation error is about 0.0002, which means that, averagely speaking, the difference between an element in M^* and the corresponding element in M at the same index is 0.0002.

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[18]We employ the “iterative scaling” method, which was first proposed by Sinkhorn (1964) for randomly generating bi-stochastic matrices and then extended to generating plane-stochastic matrices by Bapat (1982), Raghavan (1984), and Franklin and Lorenz (1989) among others.

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