Random Assignments on Preference Domains with a Tier Structure*

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Abstract

We address a standard random assignment problem (Bogomolnaia and Moulin, 2001). A weakly connected domain admitting an sd-strategy-proof, sd-efficient and equal-treatment-of-equals rule is characterized to be a restricted tier domain. Conversely, on such a domain, the probabilistic serial rule is uniquely characterized by either sd-strategy-proofness, sd-efficiency and equal treatment of equals, or sd-efficiency and sd-envy-freeness. Moreover, we provide an algorithm to construct unions of multiple restricted tier domains, each of which admits an sd-strategy-proof, sd-efficient and sd-envy-free rule.

Keywords: sd-strategy-proofness; sd-efficiency; equal treatment of equals; sd-envy-freeness; probabilistic serial rule; restricted tier domains

JEL Classification: C78, D71.

1 Introduction

We consider the problem of allocating several indivisible objects to a group of agents, each of whom receives at most one object.[1] Each agent reports a strict ordinal preference on objects to the planner, and then the planner assigns a lottery over objects to each agent. The profile of lotteries agents receive is called a *random assignment*. To extend a preference on objects to an assessment on lotteries, the *stochastic dominance extension* introduced by Gibbard (1977) is widely

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[1]Classical examples include assigning college seats to applicants (Gale and Shapley, 1962), houses to residents (Shapley and Scarf, 1974), and jobs to workers (Hylland and Zeckhauser, 1979). Also see papers of Svensson (1999), Pápai (2000), Ehlers (2002) and Pycia and Ünver (2017).

adopted: A lottery is viewed at least as good as another one if the former (first-order) stochastically dominates the latter according to the ordinal preference over objects.[2] Equivalently, under the von-Neumann-Morgenstern hypothesis, a lottery stochastically dominates another one if and only if it delivers an expected utility weakly higher than that delivered by the opponent for *every* cardinal utility representing the ordinal preference.

With the stochastic dominance extension, several axioms are defined for designing random assignment rules which associate each profile of reported preferences to a random assignment. First, *sd-efficiency* requires that no reassignment can be arranged such that all agents are at least as well as before, and someone receives a strictly better lottery. Second, random assignment rules should provide incentives for agents to truthfully reveal their preferences. Accordingly, *sd-strategy-proofness* is introduced, saying that for each agent, the lottery delivered by truth-telling stochastically dominates the lottery induced by any preference misrepresentation, regardless of others' preferences. In addition, ex ante fairness in the sense of either *equal treatment of equals* or *sd-envy-freeness* is imposed. As suggested by the names, *equal treatment of equals* requires that agents reporting the same preferences receive the same lottery, while *sd-envy-freeness* is stronger, and requires that an agent always weakly prefers her own lottery to others'.

Two classic random assignment rules have been widely studied in the literature: the *random serial dictatorship rule* (Abdulkadiroğlu and Sönmez, 1998) and the *probabilistic serial rule* (Crès and Moulin, 2001; Bogomolnaia and Moulin, 2001). On the one hand, the random serial dictatorship rule is *sd-strategy-proof* and *equal-treatment-of-equals*, but not *sd-efficient* (see Abdulkadiroğlu and Sönmez, 2003; Kesten, 2009). On the other hand, the probabilistic serial rule is *sd-efficient* and *sd-envy-free*, but fails *sd-strategy-proofness*. Moreover, an impossibility result has been established by Bogomolnaia and Moulin (2001): When the numbers of objects and agents are identical and at least four, and agents' preferences are from the *universal domain* with no restriction, no random assignment rule satisfies *sd-strategy-proofness*, *sd-efficiency* and *equal treatment of equals*. Recently, this impossibility has also been established on some restricted preference domains, e.g., single-peaked domains and single-dipped domains (see Kasajima, 2013; Altuntaş, 2016; Chang and Chun, 2017).

These impossibilities raise a natural question: Is there a reasonable restricted preference domain which admits an *sd-strategy-proof*, *sd-efficient* and *equal-treatment-of-equals* random assignment rule? Furthermore, if the answer is in the affirmative, what are the admissible random assignment rules? This paper provides answers to these two questions.

We execute our investigation in a class of rich domains, *weakly connected* domains, which occupies a prominent position in the literature (see Remark 1). Two preferences are called *neighbors* if across these two preferences, several pair(s) of contiguously ranked objects are locally switched, and all other objects are identically ranked. A domain is then said *weakly connected* if any two distinct preferences are connected via a sequence of preferences in the domain which are consecutively neighbored. This implies that the difference of any two preferences in a *weakly connected* domain can be reconciled via a sequence of local switchings. Our first main result (Theorem 1) shows that a *weakly connected* domain admitting an *sd-strategy-proof*, *sd-efficient*, and *equal-treatment-of-equals* rule must be a *restricted tier domain*. To construct a *restricted tier domain*, objects are first partitioned into several blocks, each of which contains one or two objects. Then, all preferences are required to respect a common ranking of blocks, referred to as a restricted tier

[2] Other preference extensions are also investigated (e.g., Aziz et al., 2014; Bogomolnaia, 2015; Cho, 2018).

structure. As an example, consider a skyscraper with two apartments on each floor. A restricted tier structure can be elicited according to floors from the top down to the bottom: All agents prefer higher apartments to lower ones. Between two apartments on the same floor, the preferences are arbitrary across agents. The second main result (Theorem 2) searches for the desirable rules on a *restricted tier domain*, and characterizes the probabilistic serial rule by either *sd-strategy-proofness*, *sd-efficiency* and *equal treatment of equals*, or *sd-efficiency* and *sd-envy-freeness*.

Logically, our domain characterization result identifies, within the class of *weakly connected* domains, the exact boundary between the ones admitting desirable rules and the ones not. Normatively, we treat it as a negative result since a *restricted tier domain* is so restrictive that agents are almost required to have the same preference. However, we believe that our domain characterization result is critically different from and more informative than all existing impossibility results alluded above. First, it implies every existing impossibility result. One can see this by simply verifying *weak connectedness* and the failure of the restricted tier structure. Second, our domain characterization result implies the nonexistence of *sd-strategy-proof*, *sd-efficient* and *equal-treatment-of-equals* rules on some important domains that have not been studied by the literature of random assignment (see Remark 1). Third, our domain characterization is potentially useful in distinguishing possibility and impossibility when one in the future studies a specific interesting assignment problem, and encounters with a particular restricted preference domain. Last, it suggests that to find a reasonable restricted domain which admits a desirable rule, we have to go beyond the *weakly connected* domains. More specifically, given an arbitrary domain (not necessarily *weakly connected*), we partition it into multiple *weakly connected* subdomains which are mutually disconnected.[3] Then, the existence of an *sd-strategy-proof*, *sd-efficient* and *equal-treatment-of-equals* rule implies that each subdomain must be a *restricted tier domain*. In other words, any domain admitting an *sd-strategy-proof*, *sd-efficient* and *equal-treatment-of-equals* rule must be a union of *restricted tier domains*. Following this direction, we provide an algorithm which gradually excludes preferences from the universal domain, and eventually generates a union of *restricted tier domains* which is not *weakly connected*. More importantly, we show that every domain generated by the algorithm is equivalent to a *sequentially dichotomous domain* of Liu (2019), and hence admits an *sd-strategy-proof*, *sd-efficient* and *sd-envy-free* rule (see Proposition 1).

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 presents the two main results. Section 4 studies the union of *restricted tier domains*, while Section 5 concludes. The Appendix gathers the omitted proofs.

2 Model

Let $A \equiv \{a, b, c, d, \dots\}$ be a finite set of objects and $I \equiv \{1, 2, \dots, n\}$ be a finite set of agents. We assume $|A| = |I| = n \geqslant 4$. Each agent i is equipped with a (strict) preference P_i over A which is complete, transitive and antisymmetric, i.e., a linear order. Given $a, b \in A$, $a\ P_i\ b$ is interpreted as "a is strictly preferred to b according to P_i", and $a\ P_i!\ b$ denotes that "a is strictly

[3] Starting from an arbitrary preference in the given domain, we repeatedly include in a subdomain preferences which are neighbors to any preference already included. When no more preference can be included, we identify a *weakly connected* subdomain. The same inclusion process can be repeatedly applied to the remaining preferences, if any, to identify all *weakly connected* subdomains. In this manner, we partition the given domain into multiple subdomains such that every subdomain is *weakly connected*, and no one preference in any subdomain is a neighbor to any preference in another subdomain.

preferred to b according to P_i, and a and b are *contiguously* ranked in P_i", i.e., $a\ P_i\ b$, and there exists no $c \in A$ such that $a\ P_i\ c$ and $c\ P_i\ b$. Let $r_k(P_i)$, $k = 1, \ldots, n$, denote the k-th ranked object in preference P_i. Let $B(P_i, a) = \{x \in A : x\ P_i\ a\}$ denote the (strict) upper contour set of a at P_i. Let $\mathbb{P}$ denote the set of *all* preferences. The set of admissible preferences is $\mathbb{D} \subseteq \mathbb{P}$, referred to as a *preference domain*. In particular, $\mathbb{P}$ is called the *universal domain*, and a proper subset of $\mathbb{P}$ is called a *restricted domain*. We assume that all agents have the same preference domain $\mathbb{D}$. A *preference profile* $P \equiv (P_1, \ldots, P_n) \equiv (P_i, P_{-i}) \in \mathbb{D}^n$ is an n-tuple of admissible preferences.

Let $\Delta(A)$ denote the set of lotteries over A. Given $\lambda \in \Delta(A)$, λ_a denotes the probability allotted to the object a. A (random) **assignment** is a bi-stochastic matrix $L \equiv [L_{ia}]_{i\in I, a\in A}$, namely a non-negative square matrix whose elements in each row and each column sum to unity respectively, i.e., (i) $L_{ia} \geqslant 0$ for all $i \in I$ and $a \in A$, (ii) $\sum_{a\in A} L_{ia} = 1$ for all $i \in I$, and (iii) $\sum_{i\in I} L_{ia} = 1$ for all $a \in A$. An element L_{ia} is interpreted as the probability of agent i receiving object a. Then, the i-th row of L, denoted L_i, specifies agent i's lottery over A. Let $\mathcal{L}$ denote the set of all bi-stochastic matrices. The Birkhoff-von-Neumann theorem states that every bi-stochastic matrix can be decomposed as a lottery over permutation matrices. Hence, a random assignment can be implemented by randomly drawing a permutation matrix from a Birkhoff-von-Neumann decomposition and then allocating deterministic objects accordingly. A (random assignment) **rule** is a mapping $\varphi : \mathbb{D}^n \to \mathcal{L}$ which specifies a random assignment at each profile of reported preferences. Given $P \in \mathbb{D}^n$, $\varphi_{ia}(P)$ denotes the probability of agent i receiving object a, and $\varphi_i(P)$ denotes the lottery assigned to agent i. For notational convenience, given a subset $B \subseteq A$, let $\varphi_{iB}(P) \equiv \sum_{a\in B} \varphi_{ia}(P)$ denote the probability of agent i receiving an object in B at profile P.

Agents assess lotteries according to (first-order) stochastic dominance. Formally, given $P_i \in \mathbb{D}$ and lotteries $\lambda, \lambda' \in \Delta(A)$, λ *stochastically dominates* λ' according to P_i, denoted $\lambda\ P_i^{sd}\ \lambda'$, if $\sum_{l=1}^{k} \lambda_{r_l(P_i)} \geqslant \sum_{l=1}^{k} \lambda'_{r_l(P_i)}$ for all $k = 1, \ldots, n$. Given $P \in \mathbb{D}^n$, an assignment L is *sd-efficient* if there exists no other $L' \in \mathcal{L}$ *Pareto dominating* L, i.e., $L' \neq L$ and $L'_i\ P_i^{sd}\ L_i$ for all $i \in I$. Accordingly, a rule $\varphi : \mathbb{D}^n \to \mathcal{L}$ is **sd-efficient** (or **sd-Eff**) if $\varphi(P)$ is *sd-efficient* for all $P \in \mathbb{D}^n$. Next, a rule is *sd-strategy-proof* if for every agent, her lottery under truth-telling always stochastically dominates her lottery induced by any misrepresentation according to her true preference, regardless of others' preference reporting. Formally, a rule $\varphi : \mathbb{D}^n \to \mathcal{L}$ is **sd-strategy-proof** (or **sd-SP**) if for all $i \in I$, $P_i, P'_i \in \mathbb{D}$ and $P_{-i} \in \mathbb{D}^{n-1}$, $\varphi_i(P_i, P_{-i})\ P_i^{sd}\ \varphi_i(P'_i, P_{-i})$. Last, for fairness, given $P \in \mathbb{D}^n$, an assignment L is said *equal-treatment-of-equals* if all agents reporting the same preference receive the same lottery, i.e., for all $i, j \in I$, $[P_i = P_j] \Rightarrow [L_i = L_j]$, and a rule $\varphi : \mathbb{D}^n \to \mathcal{L}$ is **equal-treatment-of-equals** (or **ETE**) if $\varphi(P)$ is *equal-treatment-of-equals* for all $P \in \mathbb{D}^n$. As a stronger notion of fairness, an assignment L is *sd-envy-free* if every agent treats her own lottery weakly better than any other's, i.e., $L_i\ P_i^{sd}\ L_j$ for all $i, j \in I$. Accordingly, a rule $\varphi : \mathbb{D}^n \to \mathcal{L}$ is **sd-envy-free** (or **sd-EF**) if $\varphi(P)$ is *sd-envy-free* for all $P \in \mathbb{D}^n$. Henceforth, we adopt the mentioned abbreviations of these four axioms throughout the paper.

2.1 The Probabilistic Serial Rule

In this section, we formally introduce an important random assignment rule, the probabilistic serial (or PS) rule, which selects a random assignment at a given preference profile by the simultaneous eating algorithm with the uniform speed. Such an algorithm hypothetically treats the objects infinitely divisible and specifies the random assignment by an iterative procedure. Starting from time 0, every agent consumes her favorite object at the uniform speed, until an object reaches its exhaustion. Then, every agent resumes consuming her favorite object in the remaining ones at the uniform speed until another object is exhausted. This procedure is repeated until all objects are exhausted. Finally, the share of an object consumed by an agent is interpreted as the probability of this agent receiving this object.

We borrow the notation of Kojima and Manea (2010) to formally define the PS rule. Given $P \in \mathbb{D}^n$, for any $a \in A' \subseteq A$, let $N(a, A') \equiv \big\{i \in I : a\ P_i\ b \text{ for all } b \in A' \backslash \{a\}\big\}$ be the set of agents whose favorite object in A' is a.

Definition 1 *The **Probabilistic Serial** rule is a mapping $PS : \mathbb{D}^n \to \mathcal{L}$, where given $P \in \mathbb{D}^n$, the random assignment $PS(P)$ is specified by the following iteration.*

Initially, let $t^0 \equiv 0$, $A^0 \equiv A$, $L^0_{ia} \equiv 0$ for all $i \in I$ and $a \in A$.

For each $v = 1, \ldots, \bar{v}$, let

$$t^v \equiv \min_{a \in A^{v-1}} \max \left\{t \in [0,1] : \sum\nolimits_{i \in I} L^{v-1}_{ia} + |N(a, A^{v-1})| \cdot (t - t^{v-1}) \leqslant 1\right\}, \tag{1}$$

$$A^v \equiv A^{v-1} \backslash \left\{a \in A^{v-1} : \sum\nolimits_{i \in I} L^{v-1}_{ia} + |N(a, A^{v-1})| \cdot (t^v - t^{v-1}) = 1\right\}, \tag{2}$$

$$L^v_{ia} \equiv \begin{cases} L^{v-1}_{ia} + (t^v - t^{v-1}) & \text{if } i \in N(a, A^{v-1}), \\ L^{v-1}_{ia} & \text{otherwise}. \end{cases} \tag{3}$$

The final step $\bar{v}$ is identified by $A^{\bar{v}} = \emptyset$ and $A^{\bar{v}-1} \neq \emptyset$. Let $PS(P) \equiv [L^{\bar{v}}_{ia}]_{i \in I,\, a \in A}$.

For each period $1 \leqslant v \leqslant \bar{v}$, t^{v-1} denotes the beginning of this period, A^{v-1} denotes the set of available objects, and $L^{v-1} \equiv [L^{v-1}_{ia}]_{i \in I, a \in A}$ denotes the cumulative assignment. The end of the v-th period t^v is determined by the earliest time when some available object reaches its exhaustion, as defined by Equation (1). Then, we update the set of available objects A^v by excluding the exhausted objects from A^{v-1}, as shown in Equation (2). Last, according to Equation (3), we update the cumulative assignment to $L^v \equiv [L^v_{ia}]_{i \in I,\, a \in A}$ by assigning to each agent $t^v - t^{v-1}$ share of her favorite object in A^{v-1}.

2.2 Weakly Connected Domains

We restrict attention to a large class of preference domains, *weakly connected* domains. Formally, two distinct preferences P_i, P'_i are called **neighbors**, denoted $P_i \approx P'_i$, if whenever two objects are oppositely ranked across P_i and P'_i, they are contiguously ranked at both P_i and P'_i, i.e.,

$$[a\ P_i\ b \text{ and } b\ P'_i\ a] \Rightarrow [a\ P_i!\ b \text{ and } b\ P'_i!\ a].$$

Note that across two neighbored preferences, no object is involved in more than one preference reversal. Therefore, two neighbored preferences differ in locally switching several pair(s) of contiguously ranked objects, i.e., there exist $\{(b_l, a_l) : l = 1, \ldots, t\}$, $t \geqslant 1$, and $1 \leqslant k_1 < k_1 + 1 < k_2 < k_2 + 1 < \cdots < k_{t-1} < k_{t-1} + 1 < k_t < n$, such that (i) $b_l = r_{k_l}(P_i) = r_{k_l+1}(P'_i)$ and $a_l = r_{k_l+1}(P_i) = r_{k_l}(P'_i)$ for all $l = 1, \ldots, t$, and (ii) $[x\ P_i\ y] \Leftrightarrow [x\ P'_i\ y]$ for all $(x, y) \notin \{(b_l, a_l) : l = 1, \ldots, t\}$. In particular, if two neighbored preferences P_i and P'_i disagree on *exactly* one contiguously ranked pair of objects, they are called **adjacent**. We provide the following example to illustrate.

Example 1 Let preferences P_1, P_2 and P_3 be specified as below.

$$P_1: \quad a \succ c \succ b \succ d$$
$$P_2: \quad a \succ b \succ c \succ d$$
$$P_3: \quad b \succ a \succ d \succ c$$

It is evident that P_1 and P_2 are adjacent, and P_2 and P_3 are neighbors, while P_1 and P_3 have no neighborhood relation. □

A domain $\mathbb{D}$ is **weakly connected** if for all distinct $P_i, P_i' \in \mathbb{D}$, the difference of P_i and P_i' can be reconciled via local switchings along a sequence of neighbored preferences in the domain, i.e., there exists a *path* $\{P_i^k\}_{k=1}^t \subseteq \mathbb{D}$ connecting P_i and P_i' such that $P_i^1 = P_i$, $P_i^t = P_i'$, and $P_i^k \approx P_i^{k+1}$ for all $k = 1, \ldots, t-1$.

Remark 1 The class of *weakly connected* domains includes various instances that are widely studied in the ordinal mechanism design literature (including random assignment models) under both the one-dimensional and multidimensional settings: the universal domain (Gibbard, 1977; Abdulkadiroğlu and Sönmez, 1998), the single-peaked domain (Moulin, 1980), the single-dipped domain (Barberà et al., 2012), maximal single-crossing domains (Saporiti, 2009), the separable domain (Le Breton and Sen, 1999), the top-separable domain (Le Breton and Weymark, 1999) and the multidimensional single-peaked domain (Barberà et al., 1993).[4] The notion of *weak connectedness* has also been extensively investigated in the literature of Condorcet domains (e.g., Monjardet, 2009; Puppe, 2018). Recently, some papers (e.g., Carroll, 2012; Sato, 2013; Cho, 2016a) study a proper subset of *weakly connected* domains, and show that, to ensure *sd-SP*, it suffices to guarantee that misreporting a preference adjacent to the sincere one is not profitable. □

3 Main Results

It is well known that there exists no *sd-SP*, *sd-Eff* and *ETE* rule on the universal domain. We in this section investigate the preference restriction which restores the compatibility of these axioms. First, we use an example to intuitively show a preference restriction, and explain how it helps to ensure *sd-SP* of the PS rule. We then formally introduce our domain restriction, and prove that it is a necessary condition for the existence of an *sd-SP*, *sd-Eff* and *ETE* rule in the class of *weakly connected* domains. Last, we characterize the PS rule on our restricted domain via either *sd-SP*, *sd-Eff* and *ETE*, or *sd-Eff* and *sd-EF*.

Example 2 Let $A \equiv \{a, b, c, d\}$. Given preference profiles $P \equiv (P_1, P_2, P_3, P_4)$ and $P' \equiv (P_1, P_2, P_3, P_4')$, we specify the PS assignments $PS(P)$ and $PS(P')$ below.

	$PS(P)$:	a	b	c	d	$PS(P')$:	a	b	c	d
$P_1: a \succ c \succ b \succ d$	1 :	1/2	0	1/4	1/4	1 :	1/3	0	5/12	1/4
$P_2: a \succ b \succ c \succ d$	2 :	1/2	0	1/4	1/4	2 :	1/3	2/9	7/36	1/4
$P_3: b \succ a \succ c \succ d$	3 :	0	1/2	1/4	1/4	3 :	0	5/9	7/36	1/4
$P_4: b \succ a \succ c \succ d$	4 :	0	1/2	1/4	1/4	4 :	1/3	2/9	7/36	1/4
$P_4': a \succ b \succ c \succ d$										

It reveals that the PS rule is not *sd-SP*: $PS_{4a}(P) + PS_{4b}(P) = \frac{1}{2} < \frac{5}{9} = PS_{4a}(P') + PS_{4b}(P')$ which says that by misreporting preference P_4', agent 4 gets a higher probability of receiving an

[4]We put all formal definitions and the detailed verification in Appendix A.

object strictly better than c in her true preference P_4. This manipulation occurs because the eating procedure of the PS rule is sensitive to deviations. In particular, agent 4's misrepresentation (from P_4 to P'_4) makes a reach its exhaustion earlier (from $\frac{1}{2}$ to $\frac{1}{3}$). Notice that agent 1 prefers c to b, while all others prefer both a and b to c. Thus, from P to P', agent 1 starts to consume c earlier (from $\frac{1}{2}$ to $\frac{1}{3}$), and hence consumes less a and b in total (from $\frac{1}{2}$ to $\frac{1}{3}$). Consequently, agent 4 together with 2 and 3 consume more a and b in total (from $\frac{1}{2}$ to $\frac{5}{9}$).

Next, we impose a tier-structure restriction on all agents' preferences: Objects a and b always occupy the top two ranking positions. Thus, preference P_1 is no longer admissible. For instance, consider two other profiles $\bar{P} \equiv (\bar{P}_1, P_2, P_3, P_4)$ and $\bar{P}' \equiv (\bar{P}_1, P_2, P_3, P'_4)$, and the corresponding PS assignments below.

	$PS(\bar{P})$:	a	b	c	d	$PS(\bar{P}')$:	a	b	c	d
$\bar{P}_1 : a \succ b \succ c \succ d$	1 :	1/2	0	1/4	1/4	1 :	1/3	1/6	1/4	1/4
$P_2 : a \succ b \succ c \succ d$	2 :	1/2	0	1/4	1/4	2 :	1/3	1/6	1/4	1/4
$P_3 : b \succ a \succ c \succ d$	3 :	0	1/2	1/4	1/4	3 :	0	1/2	1/4	1/4
$P_4 : b \succ a \succ c \succ d$	4 :	0	1/2	1/4	1/4	4 :	1/3	1/6	1/4	1/4
$P'_4 : a \succ b \succ c \succ d$										

It turns out that agent 4's misrepresentation is no longer profitable. Due to this particular tier structure, the combined probability of a and b assigned to agent 4 is fixed to $\frac{1}{2}$ at both $\bar{P}'$ and $\bar{P}$. Consequently, the switch of a and b across P_4 and P'_4 makes agent 4 worse off as he consumes less of b at $\bar{P}'$, i.e., $PS_{4b}(\bar{P}') = \frac{1}{6} < \frac{1}{2} = PS_{4b}(\bar{P})$. We therefore assert that the PS rule becomes *sd-SP*. □

Now, we formally introduce our preference restrictions. Let $\mathcal{P} \equiv (A_k)_{k=1}^{T}$ denote a **tier structure**, i.e., (i) *block* $A_k \subseteq A$ is nonempty for all $k = 1, \dots, T$, (ii) $A_k \cap A_{k'} = \emptyset$ for all $k \neq k'$, and (iii) $\cup_{k=1}^{T} A_k = A$. Next, we impose an additional restriction to define a **restricted tier structure**: Every block contains *at most* two objects, i.e., $1 \leqslant |A_k| \leqslant 2$ for all $k = 1, \dots, T$. Then, we establish a *(restricted) tier domain* by requiring that the order of blocks in a tier structure be embedded in all preferences.

Definition 2 *A domain $\mathbb{D}$ is a* ***tier domain*** *if there exists a tier structure $\mathcal{P} \equiv (A_k)_{k=1}^{T}$ such that for all $P_i \in \mathbb{D}$ and $a, b \in A$, $[a \in A_k, b \in A_{k'}$ and $k < k'] \Rightarrow [a\ P_i\ b]$. Let $\mathbb{D}(\mathcal{P})$ denote the tier domain containing all admissible preferences. In particular, $\mathbb{D} \subseteq \mathbb{D}(\mathcal{P})$ is a* ***restricted tier domain*** *if $\mathcal{P}$ is a restricted tier structure.*

Note that all preferences of a *restricted tier domain* are pairwise neighbors. Hence, each *restricted tier domain* is *weakly connected*.

Remark 2 In an auction model, Bikhchandani et al. (2006) study a class of domains with a particular *tier structure*, the order-based domains, where all (quasi linear) cardinal utilities induce an identical ordinal preference on objects at each payment level. More recently, domains with *tier structures* are also investigated in two-sided matchings (e.g., Akahoshi, 2014; Kandori et al., 2010), school choice (e.g., Kesten, 2010; Kesten and Kurino, 2013), and spectrum license auctions (Zhou and Serizawa, 2018). □

Remark 3 Notice that the restricted tier structure is a straightforward instance satisfying the value-restriction condition of Sen (1966), which is a sufficient condition for majority voting to be well-defined. A domain is *value-restricted* if for any three objects, one of them is *never* ranked either the best (among these three objects) in all preferences, or the medium in all preferences, or the worst in all preferences. □

Now, we present our first main result.

Theorem 1 *If a weakly connected domain admits an sd-SP, sd-Eff and ETE rule, it is a restricted tier domain.*

Proof: We prove Theorem 1 by two lemmas. Lemma 1 identifies two independent properties of an arbitrary domain (unnecessarily *weakly connected*), each of which implies the nonexistence of an *sd-SP*, *sd-Eff* and *ETE* rule. The first is called *the local elevating property*, while the second is called *the double elevating property*. Lemma 2 then pins down the restricted tier structure embedded in a *weakly connected* domain by cautiously avoiding the both properties.

We first introduce *the local elevating property* by a table of three preferences.

Ranking:			k		$k+1$		$k+2$	
$\bar{P}_i$:	$\underbrace{\cdots\cdots}_{B(\bar{P}_i,a)=B(P_i,a)}$	$\succ$	a	$\succ$	c	$\succ$	b	$\succ \cdots\cdots\cdots$
P_i:	$\overbrace{\underbrace{\cdots\cdots}_{B(P_i,a)=B(\hat{P}_i,b)}}$	$\succ$	a	$\succ$	b	$\succ$	c	$\succ \cdots\cdots\cdots$
$\hat{P}_i$:	$\overbrace{\cdots\cdots}$	$\succ$	b	$\succ$	a	$\succ$	c	$\succ \cdots\cdots\cdots$

Table 1: The local elevating property

Observe first that, $\bar{P}_i$, P_i and $\hat{P}_i$ share an identical set of top $(k-1)$-ranked objects, and may differ in the rankings inside the identical set. Second, in all three preferences of Table 1, three objects a, b and c cluster in three consecutive ranking positions. Last, object b takes three distinct positions while the relative ranking between a and c is fixed. From $\bar{P}_i$ to P_i, object b is raised from the $(k+2)$-th position to the $(k+1)$-th position by locally overtaking c, while from P_i to $\hat{P}_i$, b is lifted one position further by locally overtaking a. Note that P_1, P_2 and P_3 of Example 2 satisfy *the local elevating property*. We formally introduce the definition of *the local elevating property* below.

Definition 3 *Domain $\mathbb{D}$ satisfies the* ***local elevating property*** *if there exist $\bar{P}_i, P_i, \hat{P}_i \in \mathbb{D}$, $a, b, c \in A$ and $1 \leqslant k \leqslant n-2$ satisfying the following two conditions:*

1. $r_k(\bar{P}_i) = a$, $r_{k+1}(\bar{P}_i) = c$, $r_{k+2}(\bar{P}_i) = b$,
   $r_k(P_i) = a$, $r_{k+1}(P_i) = b$, $r_{k+2}(P_i) = c$,
   $r_k(\hat{P}_i) = b$, $r_{k+1}(\hat{P}_i) = a$, $r_{k+2}(\hat{P}_i) = c$, *and*
2. $B(\bar{P}_i, a) = B(P_i, a) = B(\hat{P}_i, b)$.

The double elevating property differs from *the local elevating property* as we introduce an additional object d which is consecutively ranked below c in P_i, and ranks above c in $\hat{P}_i$. Consequently, besides the same local elevating process of object b in Table 1, one would observe an additional elevating process in the opposite direction: Object c overtakes d from the $(k+3)$-th position at $\hat{P}_i$ to the $(k+2)$-th position at P_i, and continues to overtake b from P_i to $\bar{P}_i$ (see Table 2 below).

			k		$k+1$		$k+2$		$k+3$		
$\bar{P}_i$:	$\cdots\cdots\cdots$	$\succ$	a	$\succ$	c	$\succ$	b	$\succ$	$\cdot$	$\succ$	$\cdots$
	$B(\bar{P}_i, a) = B(P_i, a)$										
P_i:	$\cdots\cdots\cdots$	$\succ$	a	$\succ$	b	$\succ$	c	$\succ$	d	$\succ$	$\cdots$
	$B(P_i, a) = B(\hat{P}_i, b)$										
$\hat{P}_i$:	$\cdots\cdots\cdots$	$\succ$	b	$\succ$	a	$\succ$	d	$\succ$	c	$\succ$	$\cdots$

Table 2: The double elevating property

Definition 4 *Domain $\mathbb{D}$ satisfies the* ***double elevating property*** *if there exist $\bar{P}_i, P_i, \hat{P}_i \in \mathbb{D}$, $a, b, c, d \in A$, and $1 \leqslant k \leqslant n-3$ satisfying the following two conditions:*

1. $r_k(\bar{P}_i) = a$, $r_{k+1}(\bar{P}_i) = c$, $r_{k+2}(\bar{P}_i) = b$,
   $r_k(P_i) = a$, $r_{k+1}(P_i) = b$, $r_{k+2}(P_i) = c$, $r_{k+3}(P_i) = d$,
   $r_k(\hat{P}_i) = b$, $r_{k+1}(\hat{P}_i) = a$, $r_{k+2}(\hat{P}_i) = d$, $r_{k+3}(\hat{P}_i) = c$, *and*
2. $B(\bar{P}_i, a) = B(P_i, a) = B(\hat{P}_i, b)$.

Lemma 1 below implies that both *the local elevating property* and *the double elevating property* are sufficient conditions for the nonexistence of an *sd-SP*, *sd-Eff* and *ETE* rule.

Lemma 1 *A domain satisfying the local elevating property or the double elevating property admits no sd-SP, sd-Eff and ETE rule.*

The proof of Lemma 1 is put in Appendix B. We provide here an outline of the proof strategy. Let domain $\mathbb{D}$ satisfy *the local elevating property*.[5] Suppose that there exists an *sd-SP*, *sd-Eff* and *ETE* rule $\varphi : \mathbb{D}^n \to \mathcal{L}$. We illustrate below how a contradiction is identified in the case where n is even.[6] We first construct the following two preference profiles, which involve only the preferences in Table 1.

- $P^{3,\frac{n}{2}} \equiv (\hat{P}_1, \ldots, \hat{P}_{\frac{n}{2}-1}, P_{\frac{n}{2}}, P_{\frac{n}{2}+1}, \ldots, P_{n-1}, \bar{P}_n)$: Agents $1, \ldots, \frac{n}{2}-1$ report preference $\hat{P}_i$, agents $\frac{n}{2}, \frac{n}{2}+1, \ldots, n-1$ report P_i, while agent n reports $\bar{P}_i$.
- $P^{4,\frac{n}{2}} \equiv (\hat{P}_1, \ldots, \hat{P}_{\frac{n}{2}-1}, \hat{P}_{\frac{n}{2}}, P_{\frac{n}{2}+1}, \ldots, P_{n-1}, \bar{P}_n)$: Agents $1, \ldots, \frac{n}{2}-1, \frac{n}{2}$ report preference $\hat{P}_i$, agents $\frac{n}{2}+1, \ldots, n-1$ report P_i, while agent n reports $\bar{P}_i$.

Note that $P^{3,\frac{n}{2}}$ and $P^{4,\frac{n}{2}}$ differ exactly in agent $\frac{n}{2}$'s preferences, i.e., $P^{3,\frac{n}{2}}_{\frac{n}{2}} = P_i$ and $P^{4,\frac{n}{2}}_{\frac{n}{2}} = \hat{P}_i$. Then, *sd-SP* requires $\varphi_{\frac{n}{2}a}(P^{3,\frac{n}{2}}) + \varphi_{\frac{n}{2}b}(P^{3,\frac{n}{2}}) = \varphi_{\frac{n}{2}b}(P^{4,\frac{n}{2}}) + \varphi_{\frac{n}{2}a}(P^{4,\frac{n}{2}})$. We will induce a contradiction where this equality does not hold. In order to do so, we investigate two sequences of preference profiles.

The first sequence starts from $(P_1, P_2, \ldots, P_{n-1}, \bar{P}_n)$, and gradually changes to $P^{3,\frac{n}{2}}$ by switching, one by one, the preferences of agents $1, 2, \ldots, \frac{n}{2}-1$ from P_i to $\hat{P}_i$. For each profile of this sequence, we characterize the probabilities of a and b. Eventually, we determine $\varphi_{\frac{n}{2}a}(P^{3,\frac{n}{2}}) + \varphi_{\frac{n}{2}b}(P^{3,\frac{n}{2}})$. The second sequence goes from $(\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_{n-1}, \bar{P}_n)$ to $P^{4,\frac{n}{2}}$, and changes, one

[5] When the domain satisfies instead the *double elevating property*, the same proof strategy applies.
[6] For the case of an odd number of agents, a similar contradiction can be identified with some extra effort.

by one, agents $n-1, n-2, \ldots, \frac{n}{2}+1$'s preferences from $\hat{P}_i$ to P_i. For each preference profile in this sequence, we also characterize the probabilities of a and b. Eventually, we determine $\varphi_{\frac{n}{2}b}(P^{4,\frac{n}{2}}) + \varphi_{\frac{n}{2}a}(P^{4,\frac{n}{2}})$, and verify that it is different from $\varphi_{\frac{n}{2}a}(P^{3,\frac{n}{2}}) + \varphi_{\frac{n}{2}b}(P^{3,\frac{n}{2}})$.

Now, according to Lemma 1, the hypothesis of Theorem 1 implies that the *weakly connected* domain in question must violate both *the local elevating* and *double elevating properties*. The next lemma utilizes the negation of both properties to elicit the embedded restricted tier structure.

Lemma 2 *A weakly connected domain avoiding both the local elevating and double elevating properties is a restricted tier domain.*

The proof of Lemma 2 is put in Appendix C. We provide here a proof outline. Let $\mathbb{D}$ be a *weakly connected* domain, and violate both *the local elevating* and *double elevating properties*. We first show that among any three distinct preferences of $\mathbb{D}$, if two pairs of them are neighbors, then all three are pairwise neighbors. Consequently, by *weak connectedness*, since every pair of distinct preferences is connected via a path, they must be neighbors. Therefore, all preferences of $\mathbb{D}$ are pairwise neighbors. This implies that $\mathbb{D}$ must be a *restricted tier domain*.

In conclusion, combining Lemmas 1 and 2, we complete the proof of Theorem 1. ■

Remark 4 Theorem 1 still holds when $|A| \neq |I|$. When $|A| > |I| \geqslant 3$, Lemma 1 holds by arbitrarily choosing $|A|-|I|$ objects as the commonly least preferred objects. When $|I| > |A| \geqslant 4$, Lemma 1 still holds by introducing $|I|-|A|$ new objects as the commonly least preferred objects. In addition, Lemma 2 holds no matter whether $|A| = |I|$ or not since it pins down the restricted tier structure using only *weak connectedness* and the negation of both *the local elevating* and *double elevating properties*. □

Remark 5 Our Lemma 1 generalizes the recent impossibility theorem of Chang and Chun (2017) which says that there is no *sd-SP*, *sd-Eff* and *ETE* rule on a domain including three particular preferences such that one object takes the bottom three ranking positions respectively while all the other objects are identically ranked. Therefore, their preference condition is in fact a special case of our *local elevating property*. The proof strategy of Chang and Chun (2017) is applicable for showing the impossibility under *the local elevating property*, but becomes invalid under *the double elevating property*. More importantly, our more stylized pattern of *the local elevating* and *double elevating properties* allows us to pin down the restricted tier structure. □

Remark 6 If we strengthen the fairness axiom from *ETE* to *sd-EF*, the proof of Lemma 1 can be significantly simplified (see Appendix D). □

The next result characterizes the PS rule as the unique desirable rule on a *restricted tier domain*.

Theorem 2 *Let $\mathbb{D}$ be a restricted tier domain. Fix a rule $\varphi : \mathbb{D}^n \to \mathcal{L}$. The following three statements are equivalent:*

(i) *φ is sd-SP, sd-Eff and ETE.*

(ii) *φ is sd-Eff and sd-EF.*

(iii) *φ is the PS rule.*

Proof: Let $\mathcal{P} \equiv (A_k)_{k=1}^T$ be a restricted tier structure, and $\mathbb{D} \subseteq \mathbb{D}(\mathcal{P})$. We prove the theorem by two steps. The first proves the equivalence between (i) and (iii), and the second shows the equivalence between (ii) and (iii). Before these two steps, we present the following fact, which observes that, due to the restricted tier structure embedded in domain $\mathbb{D}$, the definition of the PS rule (Definition 1) is significantly simplified.

Fact 1 *Given $P \in \mathbb{D}^n$, an assignment $L = PS(P)$ if and only if the following two conditions hold for every block A_k, $k = 1, \dots, T$:*

1. *If $A_k \equiv \{a\}$, we have $L_{ia} = \frac{1}{n}$ for all $i \in I$.*
2. *If $A_k \equiv \{a, b\}$, let $I_k \equiv \{i \in I : a \;\; P_i \;\; b\}$, and we have L over A_k specified below:*

$$\left[|I_k| \geqslant \frac{n}{2}\right] \Rightarrow \begin{bmatrix} & a & b \\ i \in I_k: & \frac{1}{|I_k|} & \frac{2}{n} - \frac{1}{|I_k|} \\ j \notin I_k: & 0 & \frac{2}{n} \end{bmatrix}, \textit{ and } \left[|I_k| \leqslant \frac{n}{2}\right] \Rightarrow \begin{bmatrix} & a & b \\ i \in I_k: & \frac{2}{n} & 0 \\ j \notin I_k: & \frac{2}{n} - \frac{1}{n-|I_k|} & \frac{1}{n-|I_k|} \end{bmatrix}.$$

Condition 1 says that if a block contains exactly one object, all agents equally share it. Condition 2 specifies the assignment of two objects in the same block, say $A_k = \{a, b\}$. The set $I_k \equiv \{i \in I : a \;\; P_i \;\; b\}$ is either the majority group (if $|I_k| \geqslant \frac{n}{2}$), or the minority group (if $|I_k| \leqslant \frac{n}{2}$). If I_k is the majority group, each agent of them consumes $\frac{1}{|I_k|}$ share of a and $\frac{2}{n} - \frac{1}{|I_k|}$ of b, while each agent of $I \backslash I_k$ (provided $I \backslash I_k \neq \emptyset$) only consumes $\frac{2}{n}$ of b. If I_k is the minority group, the symmetric case applies.

To see that the random assignment specified above is exactly the PS assignment of Definition 1, recall the eating procedure applied on a profile P of a *restricted tier domain*. If the top ranked block contains exactly one object, every agent will consume $\frac{1}{n}$ share of it. Otherwise, let the top ranked block $A_1 \equiv \{a, b\}$. Then, all agents of $I_1 \equiv \{i \in I : a \;\; P_i \;\; b\}$ will consume a, and others will consume b until one object of A_1 reaches its exhaustion, or both reach the exhaustion simultaneously. Clearly, which object will be exhausted earlier depends on the relative size of I_1 and $I \backslash I_1$. Specifically, according to Definition 1, we have the following observations:

- If $|I_1| = \frac{n}{2}$, then $t^1 = \frac{2}{n}$, $A^1 = A \backslash \{a, b\}$ and $L^1 = \begin{bmatrix} & a & b & A\backslash\{a,b\} \\ i \in I_1: & \frac{2}{n} & 0 & 0 \\ j \notin I_1: & 0 & \frac{2}{n} & 0 \end{bmatrix}$.
- If $|I_1| > \frac{n}{2}$, then $t^1 = \frac{1}{|I_1|}$, $A^1 = A \backslash \{a\}$ and $L^1 = \begin{bmatrix} & a & b & A\backslash\{a,b\} \\ i \in I_1: & \frac{1}{|I_1|} & 0 & 0 \\ j \notin I_1: & 0 & \frac{1}{|I_1|} & 0 \end{bmatrix}$.
- If $|I_1| < \frac{n}{2}$, then $t^1 = \frac{1}{n-|I_1|}$, $A^1 = A \backslash \{b\}$ and $L^1 = \begin{bmatrix} & a & b & A\backslash\{a,b\} \\ i \in I_1: & \frac{1}{n-|I_1|} & 0 & 0 \\ j \notin I_1: & 0 & \frac{1}{n-|I_1|} & 0 \end{bmatrix}$.

In words, if $t^1 = \frac{2}{n}$, then the whole block A_1 is exhausted at time t^1. If $t^1 \neq \frac{2}{n}$, all agents will consume the remaining object in A_1 after time t^1 until the exhaustion of the whole block A_1. Then, according to Definition 1, we have $t^2 = \frac{2}{n}$ and $A^2 = A\backslash\{a,b\}$, and update the assignment to L^2 below:

$$\left[|I_1| > \frac{n}{2}\right] \Rightarrow\; L^2 = \begin{bmatrix} & a & b & A\backslash\{a,b\} \\ i \in I_1: & \frac{1}{|I_1|} & \frac{2}{n} - \frac{1}{|I_1|} & 0 \\ j \notin I_1: & 0 & \frac{2}{n} & 0 \end{bmatrix}, \text{ and}$$

$$\left[|I_1| < \frac{n}{2}\right] \Rightarrow\; L^2 = \begin{bmatrix} & a & b & A\backslash\{a,b\} \\ i \in I_1: & \frac{2}{n} & 0 & 0 \\ j \notin I_1: & \frac{2}{n} - \frac{1}{n-|I_1|} & \frac{1}{n-|I_1|} & 0 \end{bmatrix}.$$

Notice that every agent consumes $\frac{2}{n}$ of a and b in combination. By examining the eating procedure on all blocks consecutively, we eventually obtain the PS assignment as specified in Fact 1. Thus, in the following two steps, we refer to Fact 1 as the definition of the PS rule.

Step 1: (i) $\Leftrightarrow$ (iii)

As shown in Example 2, it is easy to verify that the PS rule is *sd-SP* on domain $\mathbb{D}$. We next show (i) $\Rightarrow$ (iii). We fix an arbitrary profile $P \equiv (P_1, P_2, \ldots, P_n) \in \mathbb{D}^n$, and show that $\varphi(P)$ is exactly the one specified in Fact 1. The proof consists of three claims below.

Claim 1 *$\varphi_{iA_k}(P) = \frac{|A_k|}{n}$ for all $i \in I$ and $k = 1, \ldots, T$.*

Fix arbitrary $\bar{P}_1 = \bar{P}_2 = \cdots = \bar{P}_n$. Let $P^0 \equiv (\bar{P}_1, \bar{P}_2, \ldots, \bar{P}_n)$. We consider the following n groups of preference profiles:

$$\begin{aligned} &\text{Group 1: } P^{\{i\}} \equiv (P_i, \bar{P}_{-i}) \text{ for each } i \in I, \\ &\vdots \\ &\text{Group } 1 \leqslant l \leqslant n\text{: } P^{\hat{I}} \equiv (P_{\hat{I}}, \bar{P}_{-\hat{I}}) \text{ for each } \hat{I} \subseteq I \text{ with } |\hat{I}| = l, \\ &\vdots \\ &\text{Group } n\text{: } P^{I} \equiv (P_1, P_2, \ldots, P_n). \end{aligned}$$

Note that for each $1 \leqslant l \leqslant n$, group l contains $\frac{n!}{l!(n-l)!}$ preference profiles, and $P^I = P$. We show that for each group $1 \leqslant l \leqslant n$ and each profile $P^{\hat{I}}$ of group l, $\varphi_{iA_k}(P^{\hat{I}}) = \frac{|A_k|}{n}$ for all $i \in I$ and $k = 1, \ldots, T$.

It is evident that *ETE* and feasibility imply $\varphi_{iA_k}(P^0) = \frac{|A_k|}{n}$ for all $i \in I$ and $k = 1, \ldots, T$. We next provide an induction hypothesis: Given $0 < l \leqslant n$, for every preference profile $P^{\bar{I}}$ of group $l-1$, we have $\varphi_{iA_k}(P^{\bar{I}}) = \frac{|A_k|}{n}$ for all $i \in I$ and $k = 1, \ldots, T$. Given an arbitrary $\hat{I} \subseteq I$, let $|\hat{I}| = l$, and we show $\varphi_{iA_k}(P^{\hat{I}}) = \frac{|A_k|}{n}$ for all $i \in I$ and $k = 1, \ldots, T$.

Note that $\hat{I} \neq \emptyset$. For notational convenience, we assume w.l.o.g. that $\hat{I} = \{1, \ldots, l\}$. Given an arbitrary $i \in \hat{I}$, let $\bar{I} = \hat{I}\backslash\{i\}$. From $P^{\hat{I}} \equiv (P_1, \ldots, P_{i-1}, P_i, P_{i+1}, \ldots, P_l, \bar{P}_{-\hat{I}})$ to $P^{\bar{I}} \equiv (P_1, \ldots, P_{i-1}, \bar{P}_i, P_{i+1}, \ldots, P_l, \bar{P}_{-\hat{I}})$, agent i unilaterally deviates from P_i to $\bar{P}_i$. Since P_i and $\bar{P}_i$ share the same ranking over all blocks $A_1, \ldots, A_T$, *sd-SP* and the induction hypothesis imply $\varphi_{iA_k}(P^{\hat{I}}) = \varphi_{iA_k}(P^{\bar{I}}) = \frac{|A_k|}{n}$ for all $k = 1, \ldots, T$. Therefore, we know $\varphi_{iA_k}(P^{\hat{I}}) = \frac{|A_k|}{n}$ for all $i \in \hat{I}$ and $k = 1, \ldots, T$. If $l = n$ (equivalently, $\hat{I} = I$), we have completed the verification of the induction hypothesis. Otherwise, since all agents of $I\backslash\hat{I}$ have the same preference, *ETE* and feasibility imply $\varphi_{jA_k}(P^{\hat{I}}) = \frac{|A_k| - l \times \frac{|A_k|}{n}}{n-l} = \frac{|A_k|}{n}$ for all $j \in I\backslash\hat{I}$ and $k = 1, \ldots, T$. Therefore,

$\varphi_{iA_k}(P^{\hat{I}}) = \frac{|A_k|}{n}$ for all $i \in I$ and $k = 1, \ldots, T$. This completes the verification of the induction hypothesis, and hence proves the claim.

Thus, for each $1 \leqslant k \leqslant T$, if A_k is a singleton set, say $A_k \equiv \{a\}$, we have $\varphi_{ia}(P) = \frac{1}{n}$ for all $i \in I$. Therefore, $\varphi(P)$ meets the first condition of Fact 1. Next, fix an arbitrary block $A_k \equiv \{a, b\}$. Let $I_k \equiv \{i \in I : a\, P_i\, b\}$ and $l \equiv |I_k|$. Assume w.l.o.g. that $l \geqslant \frac{n}{2}$. The verification related to $l \leqslant \frac{n}{2}$ is symmetric, and we hence omit it.

Claim 2 $\varphi_{ja}(P) = 0$ *and* $\varphi_{jb}(P) = \frac{2}{n}$ *for all* $j \in I \backslash I_k$.

If $l = n$, then $I \backslash I_k = \emptyset$, and the claim holds vacuously. Next, assume $l < n$. Thus, $I \backslash I_k \neq \emptyset$. Suppose that there exists $j^* \in I \backslash I_k$ such that $\varphi_{j^*a}(P) > 0$. Since each agent of I_k prefers a to b, *sd-Eff* implies $\varphi_{ib}(P) = 0$ for all $i \in I_k$. Then, feasibility implies $\sum_{j \in I \backslash I_k} \varphi_{jb}(P) = 1$. Thus, there are two cases to consider: (i) There exists $j \in I \backslash I_k$ such that $\varphi_{jb}(P) > \frac{1}{n-l}$, and (ii) $\varphi_{jb}(P) = \frac{1}{n-l}$ for all $j \in I \backslash I_k$. For case (i), $\varphi_{jA_k}(P) \equiv \varphi_{ja}(P) + \varphi_{jb}(P) > \frac{1}{n-l} \geqslant \frac{2}{n}$ which contradicts Claim 1. For case (ii), $\varphi_{j^*A_k}(P) \equiv \varphi_{j^*a}(P) + \varphi_{j^*b}(P) > \frac{1}{n-l} \geqslant \frac{2}{n}$ which also contradicts Claim 1. Therefore, $\varphi_{ja}(P) = 0$ for all $j \in I \backslash I_k$. Then, by Claim 1, we have $\varphi_{jb}(P) = \frac{2}{n}$ for all $j \in I \backslash I_k$. This completes the verification of the claim.

Claim 3 $\varphi_{ia}(P) = \frac{1}{l}$ *and* $\varphi_{ib}(P) = \frac{2}{n} - \frac{1}{l}$ *for all* $i \in I_k$.

Fix an arbitrary preference $\bar{P}_i \in \mathbb{D}$ with $a\ \bar{P}_i\ b$. We first construct preference profile $P^0 \equiv (\bar{P}_{I_k}, P_{-I_k})$ where every agent of I_k reports preference $\bar{P}_i$, and every agent $j \in I \backslash I_k$ reports preference P_j in profile P. To prove the claim, we consider the following l group of preference profiles:

$$\text{Group 1: } P^{\{i\}} \equiv (P_i, \bar{P}_{I_k \backslash \{i\}}, P_{-I_k}) \text{ for each } i \in I_k$$
$$\vdots$$
$$\text{Group } 1 \leqslant m \leqslant l\text{: } P^{\hat{I}} \equiv (P_{\hat{I}}, \bar{P}_{I_k \backslash \hat{I}}, P_{-I_k}) \text{ for each } \hat{I} \subseteq I_k \text{ with } |\hat{I}| = m,$$
$$\vdots$$
$$\text{Group } l\text{: } P^{I_k} \equiv (P_{I_k}, P_{-I_k}).$$

Note that for each $1 \leqslant m \leqslant l$, group m contains $\frac{l!}{m!(l-m)!}$ preference profiles, and $P^{I_k} = P$. We show that for each group $1 \leqslant m \leqslant l$ and each profile $P^{\hat{I}}$ of group m, $\varphi_{ia}(P^{\hat{I}}) = \frac{1}{l}$ and $\varphi_{ib}(P^{\hat{I}}) = \frac{2}{n} - \frac{1}{l}$ for all $i \in I_k$.

First, similar to Claim 2, we have $\varphi_{ja}(P^0) = 0$ and $\varphi_{jb}(P^0) = \frac{2}{n}$ for all $j \in I \backslash I_k$. Then, *ETE* and feasibility imply $\varphi_{ia}(P^0) = \frac{1}{l}$ and $\varphi_{ib}(P^0) = \frac{1 - \frac{2}{n} \times (n-l)}{l} = \frac{2}{n} - \frac{1}{l}$ for all $i \in I_k$. Next, we adopt an induction hypothesis: Given $0 < m \leqslant l$, for every preference profile $P^{\hat{I}}$ of group $m - 1$, we have $\varphi_{ia}(P^{\hat{I}}) = \frac{1}{l}$ and $\varphi_{ib}(P^{\hat{I}}) = \frac{2}{n} - \frac{1}{l}$ for all $i \in I_k$. Given an arbitrary $\hat{I} \subseteq I_k$, let $|\hat{I}| = m$, and we show $\varphi_{ia}(P^{\hat{I}}) = \frac{1}{l}$ and $\varphi_{ib}(P^{\hat{I}}) = \frac{2}{n} - \frac{1}{l}$ for all $i \in I_k$.

Note that $\hat{I} \neq \emptyset$. For notational convenience, we assume w.l.o.g. that $\hat{I} = \{1, \ldots, m\}$. Given an arbitrary $i \in \hat{I}$, let $\bar{I} = \hat{I} \backslash \{i\}$. From $P^{\hat{I}} \equiv (P_1, \ldots, P_{i-1}, P_i, P_{i+1}, \ldots, P_m, \bar{P}_{I_k \backslash \hat{I}}, P_{-I_k})$ to $P^{\bar{I}} \equiv (P_1, \ldots, P_{i-1}, \bar{P}_i, P_{i+1}, \ldots, P_m, \bar{P}_{I_k \backslash \hat{I}}, P_{-I_k})$, agent i unilaterally deviates from P_i to $\bar{P}_i$. Since P_i and $\bar{P}_i$ share the same ranking over all blocks $A_1, \ldots, A_T$, and both rank a over b, *sd-SP* and the induction hypothesis imply $\varphi_{ia}(P^{\hat{I}}) = \varphi_{ia}(P^{\bar{I}}) = \frac{1}{l}$ and $\varphi_{ib}(P^{\hat{I}}) = \varphi_{ib}(P^{\bar{I}}) = \frac{2}{n} - \frac{1}{l}$. Therefore, we know $\varphi_{ia}(P^{\hat{I}}) = \frac{1}{l}$ and $\varphi_{ib}(P^{\hat{I}}) = \frac{2}{n} - \frac{1}{l}$ for all $i \in \hat{I}$. If $m = l$ (equivalently,

$\hat{I} = I_k$), we have completed the verification of the induction hypothesis. Otherwise, we consider agents of $I_k \backslash \hat{I}$ at profile $P^{\hat{I}}$. Similar to Claim 2, we know $\varphi_{ja}(P^{\hat{I}}) = 0$ and $\varphi_{jb}(P^{\hat{I}}) = \frac{2}{n}$ for all $j \in I \backslash I_k$. Then, *ETE* and feasibility imply $\varphi_{ja}(P^{\hat{I}}) = \frac{1-m\times\frac{1}{l}}{l-m} = \frac{1}{l}$ and $\varphi_{jb}(P^{\hat{I}}) = \frac{1-m\times(\frac{2}{n}-\frac{1}{l})-(n-l)\times\frac{2}{n}}{l-m} = \frac{2}{n} - \frac{1}{l}$ for all $j \in I_k \backslash \hat{I}$. Therefore, $\varphi_{ia}(P^{\hat{I}}) = \frac{1}{l}$ and $\varphi_{ib}(P^{\hat{I}}) = \frac{2}{n} - \frac{1}{l}$ for all $i \in I_k$. This completes the verification of the induction hypothesis, and hence proves the claim.

Thus, by Claims 2 and 3, $\varphi(P)$ satisfies the second condition of Fact 1, as required. This completes the verification of Step 1.

Step 2: (ii) $\Leftrightarrow$ (iii)

By Bogomolnaia and Moulin (2001), we know that the PS assignment at any preference profile is *sd-Eff* and *sd-EF*. We show (ii) $\Rightarrow$ (iii).[7]

Fix an arbitrary profile $P \in \mathbb{D}^n$. First, since all preferences of P share the same rankings over all blocks $A_1, \ldots, A_T$, *sd-EF* and feasibility imply $\varphi_{iA_k}(P) = \frac{|A_k|}{n}$ for all $i \in I$ and $k = 1, \ldots, T$. Thus, for each $1 \leqslant k \leqslant T$, if A_k is a singleton set, say $A_k \equiv \{a\}$, we have $\varphi_{ia}(P) = \frac{1}{n}$ for all $i \in I$. Therefore, $\varphi(P)$ meets condition 1 of Fact 1.

Second, fix an arbitrary block $A_k \equiv \{a, b\}$. Given $I_k \equiv \{i \in I : a\ P_i\ b\}$, we assume w.l.o.g. that $|I_k| = l \geqslant \frac{n}{2}$. The verification related to $|I_k| = l \leqslant \frac{n}{2}$ is symmetric, and we hence omit it. If $l = n$, then all agents prefer a to b, and *sd-EF* and feasibility imply $\varphi_{ia}(P) = \frac{1}{n} \equiv \frac{1}{l}$ and $\varphi_{ib}(P) = \frac{1}{n} \equiv \frac{2}{n} - \frac{1}{l}$ for all $i \in I$, which meet condition 2 of Fact 1. Furthermore, assume $l < n$. We assert $\varphi_{ja}(P) = 0$ for all $j \in I \backslash I_k$. Suppose not, i.e., there exists $j^* \in I \backslash I_k$ such that $\varphi_{j^*a}(P) > 0$. Since all agents of I_k prefer a to b, *sd-Eff* implies $\varphi_{ib}(P) = 0$ for all $i \in I_k$. Then, *sd-EF* and feasibility imply $\varphi_{j^*b}(P) = \frac{1}{N-l}$. Consequently, $\varphi_{j^*A_k}(P) \equiv \varphi_{j^*a}(P) + \varphi_{j^*b}(P) > \frac{1}{N-l} \geqslant \frac{2}{n}$. Contradiction! Therefore, $\varphi_{ja}(P) = 0$ for all $j \in I \backslash I_k$. Then, *sd-EF* and feasibility imply $\varphi_{ia}(P) = \frac{1}{l}$ for all $i \in I_k$. Last, since $\varphi_{ia}(P) + \varphi_{ib}(P) = \frac{2}{n}$ for all $i \in I$, we have $\varphi_{ib}(P) = \frac{2}{n} - \frac{1}{l}$ for all $i \in I_k$, and $\varphi_{jb}(P) = \frac{2}{n}$ for all $j \in I \backslash I_k$. This proves the second condition of Fact 1 on $\varphi(P)$, and hence completes the verification of Step 2. ■

Remark 7 Theorem 2 still holds when $|I| \neq |A|$ and agents have outside options.[8] In particular, a preference is defined as a linear order on $A \cup \{\emptyset\}$ where $\emptyset$ denotes an outside option, and we extend the definition of *restricted tier domains* in the following sense: (i) An admissible preference treats a certain number of top blocks as acceptable, i.e., better than the outside option, and (ii) an admissible preference is only required to respect a restricted tier structure over its acceptable blocks. This extended notion of *restricted tier domains* strictly nests the preference domain studied by Bogomolnaia and Moulin (2002), and therefore the extension of Theorem 2 implies their characterization results. □

Remark 8 We notice that the random serial dictatorship rule remains *sd-inefficient* on a *restricted tier domain*. For instance, at the profile $\bar{P}$ of Example 2, the PS assignment Pareto dominates the

[7] For simplicity, we provide a direct proof. Alternatively, due to the restricted tier structure, we observe that each preference profile satisfies the Type 1 and 3 decomposition conditions of Cho (2016b). Then, Theorem 3 of Cho (2016b) implies that the PS assignment at each preference preference is the unique one satisfying *sd-Eff* and *sd-EF*.

[8] See the details in the working paper version which is available at http://ink.library.smu.edu.sg/soe_research/1860/.

random serial dictatorship assignment.

$PS(\bar{P})$:	a	b	c	d
1 :	1/2	0	1/4	1/4
2 :	1/2	0	1/4	1/4
3 :	0	1/2	1/4	1/4
4 :	0	1/2	1/4	1/4

$RSD(\bar{P})$:	a	b	c	d
1 :	5/12	1/12	1/4	1/4
2 :	5/12	1/12	1/4	1/4
3 :	1/12	5/12	1/4	1/4
4 :	1/12	5/12	1/4	1/4

□

4 An Extension: Beyond Weak Connectedness

In this section, we investigate domains which are not *weakly connected*, and admit an *sd-SP*, *sd-Eff* and *ETE* rule. As mentioned in the introduction, Theorem 1 implies that any domain (not necessarily *weakly connected*) admitting an *sd-SP*, *sd-Eff* and *ETE* rule must be a union of multiple *restricted tier domains*. Moreover, Lemma 1 implies that both *the local elevating* and *double elevating properties* must be violated. We introduce the notion of dichotomous refinement which systematically excludes preferences in order to reduce the instances of *the local elevating property*. We then provide an algorithm which repeatedly applies dichotomous refinements, and eventually generates a union of multiple *restricted tier domains* which completely avoids both *the local elevating* and *double elevating properties* (see Lemma 3). More importantly, we show that every domain generated by the algorithm is equivalent to a *sequentially dichotomous domain* of Liu (2019), and therefore restores *sd-SP* on the PS rule (see Proposition 1).

Given a tier structure $\mathcal{P} = (A_1, \ldots, A_{t-1}, A_t, A_{t+1}, \ldots, A_T)$, we say that two tier structures $\overline{\mathcal{P}}$ and $\underline{\mathcal{P}}$ are the **dichotomous refinements** of $\mathcal{P}$ if exactly one block A_t breaks into two nonempty subsets A_t^1 and A_t^2, i.e., $A_t^1 \cap A_t^2 = \emptyset$ and $A_t^1 \cup A_t^2 = A_t$, such that

$$\overline{\mathcal{P}} = (A_1, \ldots, A_{t-1}, A_t^1, A_t^2, A_{t+1}, \ldots, A_T) \text{ and } \underline{\mathcal{P}} = (A_1, \ldots, A_{t-1}, A_t^2, A_t^1, A_{t+1}, \ldots, A_T).$$

In particular, when $|A_t| \geqslant 3$, via dichotomous refinements, some preferences in the original *tier domain* $\mathbb{D}(\mathcal{P})$ are excluded from the refined tier domains $\mathbb{D}(\overline{\mathcal{P}})$ and $\mathbb{D}(\underline{\mathcal{P}})$, which turn out to reduce instances of *the local elevating property*. We present in the following example to illustrate. One can easily see that the dichotomous refinements also reduce instances of *the double elevating property* when $|A_t| \geqslant 4$.

Example 3 Let $A \equiv \{a, b, c, d\}$ and $\mathcal{P} \equiv (\{a\}, \{b, c, d\})$ be a tier structure. We break the block $\{b, c, d\}$ into $\{b, c\}$ and $\{d\}$, and induce the dichotomous refinements $\overline{\mathcal{P}} = (\{a\}, \{b, c\}, \{d\})$ and $\underline{\mathcal{P}} = (\{a\}, \{d\}, \{b, c\})$. When $\mathbb{D}(\mathcal{P})$ shrinks to $\mathbb{D}(\overline{\mathcal{P}}) \cup \mathbb{D}(\underline{\mathcal{P}})$, the instance of *the local elevating property* below which appeared in $\mathbb{D}(\mathcal{P})$ is eliminated because $\bar{P}_i$ is excluded.

$$\bar{P}_i: \quad a \succ c \succ d \succ b$$
$$P_i: \quad a \succ c \succ b \succ d$$
$$\hat{P}_i: \quad a \succ b \succ c \succ d$$

□

Now, we present the algorithm to repeatedly exclude preferences via a sequence of dichotomous refinements, and eventually generate a union of *restricted tier domains*. We first introduce the notation used in the algorithm. Given a tier structure $\mathcal{P} = (A_1, \dots, A_T)$, let $\mathbf{A} = \{A_1, \dots, A_T\}$ denote the *partition* of A which collects all blocks in $\mathcal{P}$.[9]

Algorithm: Initially, set $\Omega_0 \equiv (A)$, $\mathbf{A}_0 \equiv \{A\}$ and $\mathbb{D}(\Omega_0) \equiv \mathbb{P}$.

Step 1. Fixing an arbitrary nonempty proper subset $\bar{A} \subset A$, let

- $\Omega_1 \equiv \big\{(\bar{A}, A\backslash\bar{A}), (A\backslash\bar{A}, \bar{A})\big\}$,
- $\mathbb{D}(\Omega_1) \equiv \cup_{\mathcal{P}\in\Omega_1}\mathbb{D}(\mathcal{P})$ denote the union of corresponding *tier domains*, and
- $\mathbf{A}_1 \equiv \{\bar{A}, A\backslash\bar{A}\}$ be the corresponding partition of A.

If there exists a block of $\mathbf{A}_1$ containing more than 2 objects, proceed to the next step. Otherwise, terminate the algorithm.

Step $k > 1$. Let $A_t \in \mathbf{A}_{k-1}$ be an arbitrary block such that $|A_t| > 2$. Break A_t into two nonempty subsets A_t^1 and A_t^2. Then, let

- $\Omega_k \equiv \cup_{\mathcal{P}\in\Omega_{k-1}} \big\{\overline{\mathcal{P}}, \underline{\mathcal{P}} :\ \overline{\mathcal{P}} \text{ and } \underline{\mathcal{P}} \text{ are the corresponding dichotomous refinements of } \mathcal{P}\big\}$,
- $\mathbb{D}(\Omega_k) \equiv \cup_{\mathcal{P}\in\Omega_k}\mathbb{D}(\mathcal{P})$ denote the union of corresponding *tier domains*, and
- $\mathbf{A}_k \equiv \{A_t^1, A_t^2\} \cup \mathbf{A}_{k-1}\backslash\{A_t\}$ denotes the corresponding partition of A which replaces $A_t \in \mathbf{A}_{k-1}$ by A_t^1 and A_t^2.

If there exists a block of $\mathbf{A}_k$ containing more than 2 objects, proceed to the next step. Otherwise, terminate the algorithm.

It is evident that the algorithm terminates in finite steps. Notice that the outcome of the algorithm varies according to the sequence of dichotomous refinements. Henceforth, we fix a sequence of dichotomous refinements, and suppose that the algorithm terminates at step K. At each step $1 \leqslant k \leqslant K$, by refining the domain from $\mathbb{D}(\Omega_{k-1})$ to $\mathbb{D}(\Omega_k)$, we reduce the number of instances exhibiting *the local elevating property*. At the termination step K, we observe that the partition $\mathbf{A}_K$ contains $K+1$ blocks, each block contains no more than 2 objects, and Ω_K collects 2^K restricted tier structures over the $K+1$ blocks. Therefore, we obtain a union of 2^K *restricted tier domains* $\mathbb{D}(\Omega_K) = \cup_{\mathcal{P}\in\Omega_K}\mathbb{D}(\mathcal{P})$. Moreover, the next lemma shows that $\mathbb{D}(\Omega_K)$ violates both *the local elevating* and *double elevating properties*.

Lemma 3 *Domain* $\mathbb{D}(\Omega_K)$ *avoids both the local elevating and double elevating properties.*

The proof of Lemma 3 is put in Appendix E.

More importantly, the following proposition shows that domain $\mathbb{D}(\Omega_K)$ admits an *sd-SP*, *sd-Eff* and *sd-EF* rule.

Proposition 1 *Domain* $\mathbb{D}(\Omega_K)$ *admits an sd-SP, sd-Eff and sd-EF rule.*

The proof of Proposition 1 is put in Appendix F.

[9]Formally, a **partition** of A is a collection of mutually exclusive and exhaustive nonempty subsets of A. Blocks are not ordered in a partition, but linearly ordered in a tier structure.

5 Conclusion

In this paper, we have shown that if a *weakly connected* domain admits an *sd-SP*, *sd-Eff* and *ETE* rule, it is a *restricted tier domain*, and this desirable rule is uniquely the PS rule. Normatively, we treat our domain characterization as a negative result, since a restricted tier structure almost requires that all agents have the same preference, and it implies the nonexistence of *sd-SP*, *sd-Eff* and *ETE* rules on almost all domains studied in the literature.

Our results help in understanding the boundary between possibilities and impossibilities on designing a desirable random assignment rule. Within *weakly connected* domains, the exact boundary is identified. Beyond *weakly connected* domains, although the exact boundary is not depicted, we present an algorithm to construct unions of *restricted tier domains*, where possibility holds.

Appendix

A Details related to Remark 1

In this appendix, we first introduce six restricted domains, and then show that the universal domain and all these six restricted domains are *weakly connected* domains. In particular, three restricted domains are defined in the one-dimensional setting: the single-peaked domain, the single-dipped domain and maximal single-crossing domains. The other three are defined in the multi-dimensional setting: the separable domain, the top-separable domain and the multidimensional single-peaked domain.

All three restricted domains in the one-dimensional setting share a common feature: Objects are exogenously arranged on a linear order $<$. For notational convenience, let $a \leqslant b$ denote either $a < b$ or $a = b$.

First, a preference P_i is **single-peaked** on $<$ if, for each pair of objects at the same side of the peak $r_1(P_i)$, the one closer to the peak in $<$ is always preferred, i.e., $[b < a \leqslant r_1(P_i)$ or $r_1(P_i) \leqslant a < b] \Rightarrow [a \; P_i \; b]$. The *single-peaked domain* is the set containing all preferences single-peaked on $<$.

Second, a single-dipped preference performs exactly opposite to a single-peaked preference. In particular, A preference P_i is **single-dipped** on $<$ if $[b < a \leqslant r_{|A|}(P_i)$ or $r_{|A|}(P_i) \leqslant a < b] \Rightarrow [b \; P_i \; a]$. The *single-dipped domain* is the set containing all preferences single-dipped on $<$.

Third, to define a single-crossing domain, an exogenous linear order $\lhd$ needs to be fixed between preferences. A domain $\mathbb{D}$ is **single-crossing** on $(<, \lhd)$ if for all $a, b \in A$ with $a < b$ and $P_i, P_i' \in \mathbb{D}$ with $P_i \lhd P_i'$, we have $[a \; P_i' \; b] \Rightarrow [a \; P_i \; b]$ and $[b \; P_i \; a] \Rightarrow [b \; P_i' \; a]$. Furthermore, a single-crossing domain is *maximal* if it has the maximal cardinality $\frac{|A| \times (|A|-1)}{2} + 1$.

According to Propositions 3 and 4 of Carroll (2012) and Propositions 4.1 and 4.2 of Sato (2013), we know that in the universal domain, the single-peaked domain and a maximal single-crossing domain, two distinct preferences are connected via a sequence of adjacent preferences. Therefore, they are all *weakly connected* domains. The single-dipped domain is also a *weakly connected* domain as the same argument for the single-peaked domain applies.

In the multidimensional setting, the object set A is assumed to have a Cartesian product structure, i.e., $A = \times_{s \in M} A^s$ where (i) $M \equiv \{1, \ldots, m\}$ is finite and $m \geqslant 2$, and (ii) for each $s \in M$, the component set A^s contains finite and at least two elements. Thus, an object can be represented by an m-tuple, i.e., $a \equiv (a^1, \ldots, a^m) \equiv (a^s, a^{-s})$.

First, a preference P_i is **separable** if for each $s \in M$, a marginal preference over all elements of the component set A^s can be independently elicited from P_i, i.e., for all $a^s, b^s \in A^s$, we have $[(a^s, x^{-s})\ P_i\ (b^s, x^{-s})$ for some $x^{-s} \in A^{-s}] \Rightarrow [(a^s, y^{-s})\ P_i\ (b^s, y^{-s})$ for all $y^{-s} \in A^{-s}]$. The *separable domain* is the set containing all separable preferences.

Second, a top-separable preference is less restricted than a separable preference. A preference P_i, say $r_1(P_i) = x \equiv (x^s)_{s\in M}$, is **top-separable** if for all $s \in M$ and $a^s, b^s \in A^s$, we have $[a^s = x^s$ and $b^s \neq x^s] \Rightarrow [(a^s, z^{-s})\ P_i\ (b^s, z^{-s})$ for all $z^{-s} \in A^{-s}]$. The *top-separable domain* is the set containing all top-separable preferences.

Last, to introduce the multidimensional single-peaked domain, an additional restriction must be imposed on the object set: For each $s \in M$, all element of A^s are exogenously arranged on a linear order $<^s$. Symmetrically, let $a^s \leqslant^s b^s$ denote either $a^s <^s b^s$ or $a^s = b^s$. Thus, all objects are located on a product of linear orders, $\times_{s\in M} <^s$. Given $x, y \in A$, let $MB(x, y) = \{a \in A : x^s \leqslant^s a^s \leqslant^s y^s$ or $y^s \leqslant^s a^s \leqslant^s x^s$ for all $s \in M\}$ denote the minimal box collecting all objects located between x and y. Now, a preference P_i is **multidimensional single-peaked** on $\times_{s\in M} <^s$ if for all $a, b \in A$, we have $\big[a \in MB\big(r_1(P_i), b\big)$ and $a \neq b\big] \Rightarrow [a\ P_i\ b]$. The *multidimensional single-peaked domain* is the set containing all admissible preferences.

Chatterji and Zeng (2019) introduce a natural generalization of adjacency in the multidimensional setting: Preferences P_i and P_i' are **adjacent⁺** if the following two conditions are satisfied:

(1) Both P_i and P_i' are separable preferences.

(2) There exist $s \in M$ and $a^s, b^s \in A^s$ such that $(a^s, z^{-s})\ P_i!\ (b^s, z^{-s})$ and $(b^s, z^{-s})\ P_i'!\ (a^s, z^{-s})$ for all $z^{-s} \in A^{-s}$, and $[x\ P_i\ y] \Leftrightarrow [x\ P_i'\ y]$ for all $(x, y) \notin \big\{\big((a^s, z^{-s}), (b^s, z^{-s})\big) : z^{-s} \in A^{-s}\big\}$.

One would note that across the adjacent⁺ pair P_i and P_i', $|A^{-s}|$ pairs of objects are locally switched, while all other objects are commonly ranked. Therefore, the notion of adjacency⁺ is a special case of our neighborhood. Chatterji and Zeng (2019) show that in each one of the three multidimensional domains, two distinct preferences are connected via a sequence of preferences such that each consecutive pair is either adjacent or adjacent⁺. Therefore, all these three multidimensional domains are *weakly connected* domains.

B Proof of Lemma 1

Suppose that $\mathbb{D}$ contains preferences $\bar{P}_i, P_i, \hat{P}_i$ of Definition 3 or 4. For notational convenience, let $B \equiv B(\bar{P}_i, a) = B(P_i, a) = B(\hat{P}_i, b)$. Let $\varphi : \mathbb{D}^n \to \mathcal{L}$ be an *sd-SP*, *sd-Eff* and *ETE* rule. Let $\bar{n} \equiv \frac{n}{2}$ if n is even, and $\bar{n} \equiv \frac{n-1}{2}$ if n is odd. We search for a contradiction. We consider six groups of preference profiles: **Profile Groups I - VI** (Table 3 below). Note that every preference profile in Table 3 consists of only preference(s) of $\bar{P}_i$, P_i and $\hat{P}_i$.

Profile Group I: n **is either even or odd**	**Profile Group II:** n **is either even or odd**
$P^{1,0} = (P_1, \ldots, P_n)$ $P^{1,m} = (\hat{P}_1, \ldots, \hat{P}_m, P_{m+1}, \ldots, P_n)$, where $m = 1, \ldots, \bar{n}$.	$P^{2,0} = (\hat{P}_1, \ldots, \hat{P}_{n-1}, \hat{P}_n)$ $P^{2,m} = (\hat{P}_1, \ldots, \hat{P}_{n-m}, P_{n-m+1}, \ldots, P_n)$, where $m = 1, \ldots, \bar{n}$.
Profile Group III: n **is either even or odd**	**Profile Group IV:** n **is either even or odd**
$P^{3,1} = (P_1, \ldots, P_{n-1}, \bar{P}_n)$ $P^{3,m} = (\hat{P}_1, \ldots, \hat{P}_{m-1}, P_m, \ldots\ldots\ldots, P_{n-1}, \bar{P}_n)$, where $m = 2, \ldots, \bar{n}, \bar{n}+1$.	$P^{4,1} = (\hat{P}_1, \ldots, \hat{P}_{n-2}, \hat{P}_{n-1}, \bar{P}_n)$ $P^{4,m} = (\hat{P}_1, \ldots, \hat{P}_{n-m}, P_{n-m+1}, \ldots\ldots\ldots, P_{n-1}, \bar{P}_n)$, where $m = 2, \ldots, \bar{n}$.
Profile Group V: n **is odd**	**Profile Group VI:** n **is odd**
$P^{5,1} = (P_1, \ldots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$ $P^{5,m} = (\hat{P}_1, \ldots, \hat{P}_{m-1}, P_m, \ldots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$, where $m = 2, \ldots, \bar{n}, \bar{n}+1$.	$P^{6,2} = (\hat{P}_1, \ldots, \hat{P}_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$ $P^{6,m} = (\hat{P}_1, \ldots, \hat{P}_{n-m}, P_{n-m+1}, \ldots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$, where $m = 3, \ldots, \bar{n}$.

Table 3: Preference Profile Groups I - VI

Claim 4 *Under both the local elevating and double elevating properties, at every profile $\tilde{P}$ of profile groups I-VI, we have $\varphi_{iB}(\tilde{P}) = \frac{k-1}{n}$ for all $i \in I$.*

Proof: The claim follows from a repeated application of *sd-SP* and *ETE*. The verification is routine, and we hence omit the detailed proof. ■

Claim 5 *Under the local elevating property, at every profile $\tilde{P}$ of profile groups I-VI, we have $\sum_{x \in \{a,b,c\}} \varphi_{ix}(\tilde{P}) = \frac{3}{n}$ for all $i \in I$.*

Proof: The claim follows from a repeated application of *sd-SP* and *ETE*. The verification is routine, and we hence omit the detailed proof. ■

Claim 6 *In profile group I, for each $m = 0, 1, \ldots, \bar{n}$, at $P^{1,m} = (\hat{P}_1, \ldots, \hat{P}_m, P_{m+1}, \ldots, P_n)$, the random assignment $\varphi(P^{1,m})$ over a, b, c and d is specified below.*

	a	b	c
$1, \ldots, m$:	0	$\frac{2}{n}$	$\frac{1}{n}$
$m+1, \ldots, n$:	$\frac{1}{n-m}$	$\frac{n-2m}{n(n-m)}$	$\frac{1}{n}$

Under the local elevating property

	a	b	c	d
$1, \ldots, m$:	0	$\frac{2}{n}$	0	$\frac{2}{n}$
$m+1, \ldots, n$:	$\frac{1}{n-m}$	$\frac{n-2m}{n(n-m)}$	$\frac{1}{n-m}$	$\frac{n-2m}{n(n-m)}$

Under the double elevating property

Proof: The proof consists of 2 steps. In the first step, we specify the random assignment over a and b at every preference profile under both *the local elevating* and *double elevating properties*. In the second step, by Claim 5, we first automatically obtain the random assignment over c under *the local elevating property*. Next, we specify the random assignment over c and d at every preference profile under *the double elevating property*.

Step 1: Under both *the local elevating* and *double elevating properties*, at $P^{1,0} = (P_1, \ldots, P_n)$, *ETE* and feasibility imply $\varphi_{ia}(P^{1,0}) = \frac{1}{n}$ and $\varphi_{ib}(P^{1,0}) = \frac{1}{n}$ for all $i \in I$. Next, we provide an induction hypothesis: Given $0 < m \leqslant \bar{n}$, for all $0 \leqslant l < m$, the random assignment $\varphi(P^{1,l})$ over a and b is specified below:

$$\begin{array}{rcc} & a & b \\ 1,\ldots,m: & 0 & \frac{2}{n} \\ m+1,\ldots,n: & \frac{1}{n-l} & \frac{n-2l}{n(n-l)} \end{array}$$

We specify the random assignment $\varphi(P^{1,m})$ over a and b to complete the verification of the induction hypothesis.

First, by the induction hypothesis, *sd-SP* implies $\varphi_{mb}(P^{1,m}) + \varphi_{ma}(P^{1,m}) = \varphi_{mb}(P^{1,m-1}) + \varphi_{ma}(P^{1,m-1}) = \frac{2}{n}$. Then, *ETE* implies $\varphi_{ib}(P^{1,m}) + \varphi_{ia}(P^{1,m}) = \frac{2}{n}$ for all $i = 1,\ldots,m$. Furthermore, by *ETE* and feasibility, we have $\varphi_{jb}(P^{1,m}) + \varphi_{ja}(P^{1,m}) = \frac{2-m\times\frac{2}{n}}{n-m} = \frac{2}{n}$ for all $j = m+1,\ldots,n$.

Next, we assert $\varphi_{ia}(P^{1,m}) = 0$ for all $i = 1,\ldots,m$. Suppose not, i.e., there exists $i^* \in \{1,\ldots,m\}$ such that $\varphi_{i^*a(P^{1,m})} > 0$. Since every agent other than $1,\ldots,m$ prefers a to b, *sd-Eff* implies $\varphi_{jb}(P^{1,m}) = 0$ for all $j = m+1,\ldots,n$. Then, *ETE* and feasibility imply $\varphi_{i^*b}(P^{1,m}) = \frac{1}{m}$. Consequently, $\frac{2}{n} = \varphi_{i^*a}(P^{1,m}) + \varphi_{i^*b}(P^{1,m}) > \frac{1}{m}$. However, since $m \leqslant \bar{n}$, one can easily show $\frac{2}{n} \leqslant \frac{1}{m}$. Contradiction! Therefore, $\varphi_{ia}(P^{1,m}) = 0$ for all $i = 1,\ldots,m$. Thus, $\varphi_{ib}(P^{1,m}) = \frac{2}{n}$ for all $i = 1,\ldots,m$. Then, by *ETE* and feasibility, we have $\varphi_{ja}(P^{1,m}) = \frac{1}{n-m}$ and $\varphi_{jb}(P^{1,m}) = \frac{1-m\times\frac{2}{n}}{n-m} = \frac{n-2m}{n(n-m)}$ for all $j = m+1,\ldots,n$. This completes the verification of the induction hypothesis and Step 1.

Step 2: Next, under *the local elevating property*, by Claim 5, we automatically obtain the random assignment over c. Under *the double elevating property*, similar to Step 1, from $P^{1,0}$ to $P^{1,\bar{n}}$ step by step, by repeatedly applying *sd-SP*, *sd-Eff*, *ETE* and feasibility, we have the random assignment over c and d. This proves the claim. ■

Claim 7 *In profile group II, for each $m = 0,1,\ldots,\bar{n}$, at $P^{2,m} = (\hat{P}_1,\ldots,\hat{P}_{n-m},P_{n-m+1},\ldots,P_n)$, the random assignment $\varphi(P^{2,m})$ over a,b,c and d is specified below.*

$$\begin{array}{rccc} & a & b & c \\ 1,\ldots,n-m: & \frac{n-2m}{n(n-m)} & \frac{1}{n-m} & \frac{1}{n} \\ n-m+1,\ldots,n: & \frac{2}{n} & 0 & \frac{1}{n} \end{array}$$

Under the local elevating property

$$\begin{array}{rcccc} & a & b & c & d \\ 1,\ldots,n-m: & \frac{n-2m}{n(n-m)} & \frac{1}{n-m} & \frac{n-2m}{n(n-m)} & \frac{1}{n-m} \\ n-m+1,\ldots,n: & \frac{2}{n} & 0 & \frac{2}{n} & 0 \end{array}$$

Under the double elevating property

Proof: The verification is symmetric to the proof of Claim 6. ■

Claim 8 *In profile group III, under both the local elevating and double elevating properties, at $P^{3,1} = (P_1,\ldots,P_{n-1},\bar{P}_n)$, the random assignment $\varphi(P^{3,1})$ over a,b and c is specified below.*

$$\begin{array}{rccc} & a & b & c \\ 1,\ldots,n-1: & \frac{1}{n} & \frac{1}{n-1} & \frac{n-2}{n(n-1)} \\ n: & \frac{1}{n} & 0 & \frac{2}{n} \end{array}$$

Proof: The proof consists of 2 steps.

Step 1: First, by Claim 6, *sd-SP* implies $\varphi_{na}(P^{3,1}) = \varphi_{na}(P^{1,0}) = \frac{1}{n}$. Next, *sd-Eff* implies $\varphi_{nb}(P^{3,1}) = 0$. Then, by *ETE* and feasibility, we have $\varphi_{ia}(P^{3,1}) = \frac{1}{n}$ and $\varphi_{ib}(P^{3,1}) = \frac{1}{n-1}$ for all $i = 1, \ldots, n-1$.

Step 2: First, by Claim 6, *sd-SP* implies $\varphi_{nc}(P^{3,1}) + \varphi_{nb}(P^{3,1}) = \varphi_{nb}(P^{1,0}) + \varphi_{nc}(P^{1,0}) = \frac{2}{n}$. Since $\varphi_{nb}(P^{3,1}) = 0$ in Step 1, we have $\varphi_{nc}(P^{3,1}) = \frac{2}{n}$. Then, by *ETE* and feasibility, we have $\varphi_{ic}(P^{3,1}) = \frac{1-\frac{2}{n}}{n-1} = \frac{n-2}{n(n-1)}$ for all $i = 1, \ldots, n-1$. ■

Claim 9 *In profile group III, for each $m = 2, \ldots, \bar{n}$ (if n is even), or $m = 2, \ldots, \bar{n}, \bar{n}+1$ (if n is odd), at $P^{3,m} = (\hat{P}_1, \ldots, \hat{P}_{m-1}, P_m, \ldots, P_{n-1}, \bar{P}_n)$, the random assignment $\varphi(P^{3,m})$ over a, b and c is specified below*

	a	b	c
$1, \ldots, m-1$:	0	$\alpha(m)$	$\frac{3}{n} - \alpha(m)$
$m, \ldots, n-1$:	$\frac{1}{n-(m-1)}$	$\frac{1-(m-1)\times\alpha(m)}{n-m}$	$\frac{3}{n} - \frac{1}{n-(m-1)} - \frac{1-(m-1)\times\alpha(m)}{n-m}$
n:	$\frac{1}{n-(m-1)}$	0	$\frac{3}{n} - \frac{1}{n-(m-1)}$

Under the local elevating property

	a	b	c
$1, \ldots, m-1$:	0	$\alpha(m)$	0
$m, \ldots, n-1$:	$\frac{1}{n-(m-1)}$	$\frac{1-(m-1)\times\alpha(m)}{n-m}$	$\frac{n-2}{n(n-m)}$
n:	$\frac{1}{n-(m-1)}$	0	$\frac{2}{n}$

Under the double elevating property

where $\alpha(m) = \frac{2n^2-(2m-1)n+1}{n(n-1)[n-(m-1)]}$.

Proof: The proof consists of 3 steps. In the first step, we specify the random assignment over a and b under both *the local elevating* and *double elevating properties*. By Claim 5, we then automatically obtain the random assignment over c under *the local elevating property*. In the second step, we make two observations on the random assignment under *the double elevating property*, while the last step specifies the random assignment over c under *the double elevating property*.

Step 1: By Claim 6, *sd-SP* implies $\varphi_{na}(P^{3,2}) = \varphi_{na}(P^{1,1}) = \frac{1}{n-1}$. Symmetrically, by Claim 8, *sd-SP* implies $\varphi_{1b}(P^{3,2}) + \varphi_{1a}(P^{3,2}) = \varphi_{1b}(P^{3,1}) + \varphi_{1a}(P^{3,1}) = \frac{1}{n} + \frac{1}{n-1}$. Next, by *sd-Eff*, we have $\varphi_{nb}(P^{3,2}) = 0$ and $\varphi_{1a}(P^{3,2}) = 0$. Thus, $\varphi_{1b}(P^{3,2}) = \frac{1}{n} + \frac{1}{n-1} = \frac{2n-1}{n(n-1)} = \alpha(2)$. Last, by *ETE* and feasibility, we have $\varphi_{ia}(P^{3,2}) = \frac{1}{n-1}$ and $\varphi_{ib}(P^{3,2}) = \frac{1-(2-1)\alpha(2)}{n-2}$ for all $i = 2, \ldots, n-1$.

Next, we adopt an induction hypothesis: Given $2 < m \leqslant \bar{n}$ (if $\bar{n} = \frac{n}{2}$), or $2 < m \leqslant \bar{n}+1$ (if $\bar{n} = \frac{n-1}{2}$), for all $2 \leqslant l < m$, we have the random assignment $\varphi(P^{3,l})$ over a and b as follows:

	a	b
$1, \ldots, l-1$:	0	$\alpha(l)$
$l, \ldots, n-1$:	$\frac{1}{n-(l-1)}$	$\frac{1-(l-1)\times\alpha(l)}{n-l}$
n:	$\frac{1}{n-(l-1)}$	0

We specify the random assignment $\varphi(P^{3,m})$ over a and b to complete the verification of the induction hypothesis.

By Claim 6, *sd-SP* implies $\varphi_{na}(P^{3,m}) = \varphi_{na}(P^{1,m-1}) = \frac{1}{n-(m-1)}$. By *sd-Eff*, we have $\varphi_{nb}(P^{3,m}) = 0$. Next, by *sd-SP* and the induction hypothesis, we have

$$\begin{aligned}
\varphi_{m-1\,b}(P^{3,m}) + \varphi_{m-1\,a}(P^{3,m}) &= \varphi_{m-1\,a}(P^{3,m-1}) + \varphi_{m-1\,b}(P^{3,m-1}) \\
&= \frac{1}{n-(m-2)} + \frac{1-(m-2)\times\alpha(m-1)}{n-(m-1)} \\
&= \frac{1}{n-(m-2)} + \frac{1-(m-2)\times\left[\frac{2n^2-[2(m-1)-1]n+1}{n(n-1)[n-(m-2)]}\right]}{n-(m-1)} \\
&= \frac{2n^2-(2m-1)n+1}{n(n-1)[n-(m-1)]} \\
&= \alpha(m).
\end{aligned}$$

Furthermore, *ETE* implies $\varphi_{ib}(P^{3,m}) + \varphi_{ia}(P^{3,m}) = \alpha(m)$ for all $i = 1, \dots, m-1$.

Last, we show $\varphi_{ia}(P^{3,m}) = 0$ for all $i = 1, \dots, m-1$. Suppose not, i.e., there exists $i^* \in \{1, \dots, m-1\}$ such that $\varphi_{i^*a}(P^{3,m}) > 0$. Since every agent other than $1, \dots, m-1$ prefers a to b, *sd-Eff* implies $\varphi_{jb}(P^{3,m}) = 0$ for all $j = m, \dots, n$. Then, *ETE* and feasibility imply $\varphi_{i^*b}(P^{3,m}) = \frac{1}{m-1}$. Consequently, $\alpha(m) = \varphi_{i^*a}(P^{3,m}) + \varphi_{i^*b}(P^{3,m}) > \frac{1}{m-1}$. However, one can easily show

$$\begin{aligned}
\frac{1}{m-1} - \alpha(m) = \frac{1}{m-1} - \frac{2n^2-(2m-1)n+1}{n(n-1)[n-(m-1)]} &= \frac{n(n-m)[n-2(m-1)]-(m-1)}{(m-1)n(n-1)[n-(m-1)]} \\
&\geqslant \begin{cases} \frac{2n^2-n+2}{2(m-1)n(n-1)[n-(m-1)]} > 0 & \text{if } n \text{ is even,} \\ \frac{(n-1)^2}{2(m-1)n(n-1)[n-(m-1)]} > 0 & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

Contradiction! Therefore, $\varphi_{ia}(P^{3,m}) = 0$ for all $i = 1, \dots, m-1$. Hence, $\varphi_{ib}(P^{3,m}) = \alpha(m)$ for all $i = 1, \dots, m-1$. Last, by *ETE* and feasibility, we have $\varphi_{ja}(P^{3,m}) = \frac{1-\frac{1}{n-(m-1)}}{n-m} = \frac{1}{n-(m-1)}$ and $\varphi_{jb}(P^{3,m}) = \frac{1-(m-1)\times\alpha(m)}{n-m}$ for all $j = m, \dots, n-1$. In conclusion, we have the random assignment $\varphi(P^{3,m})$ over a and b as follows:

	a	b
$1, \dots, m-1$:	0	$\alpha(m)$
$m, \dots, n-1$:	$\frac{1}{n-(m-1)}$	$\frac{1-(m-1)\alpha(m)}{n-m}$
n:	$\frac{1}{n-(m-1)}$	0

This completes the verification of the induction hypothesis and Step 1.

Step 2: Under *the local elevating property*, by Claim 5, we have the random assignment over c at each preference profile. We focus on specifying the random assignment over c under *the double elevating property*. We first make two observations: For each $m = 2, \dots, \bar{n}$ (if n is even), or $m = 2, \dots, \bar{n}, \bar{n}+1$ (if n is odd), at $P^{3,m} = (\hat{P}_1, \dots, \hat{P}_{m-1}, P_m, \dots, P_{n-1}, \bar{P}_n)$,

OBSERVATION 1. if $r_{k+3}(\bar{P}_i) = d$, we have $\sum_{x\in\{a,b,c,d\}} \varphi_{ix}(P^{3,m}) = \frac{4}{n}$ for all $i \in I$.

OBSERVATION 2. if $r_{k+3}(\bar{P}_i) \neq d$, we have $\varphi_{ic}(P^{3,m}) + \varphi_{id}(P^{3,m}) = \frac{2}{n}$ for all $i \in I$.

The first observation is similar to Claim 5. We prove the second observation. Let $r_{k+3}(\bar{P}_i) \neq d$. Consequently, $\varphi_{nd}(P^{3,2}) = 0$ by *sd-Eff*. Next, by Claim 6, *sd-SP* implies $\varphi_{nc}(P^{3,2}) + \varphi_{nb}(P^{3,2}) = \varphi_{nb}(P^{1,1}) + \varphi_{nc}(P^{1,1}) = \frac{2}{n}$. Since $\varphi_{nb}(P^{3,2}) = 0$ in Step 1, we have $\varphi_{nc}(P^{3,2}) = \frac{2}{n}$, and hence $\varphi_{nc}(P^{3,2}) + \varphi_{nd}(P^{3,2}) = \frac{2}{n}$. Next, recall profile $P^{3,1} = (P_1, \dots, P_{n-1}, \bar{P}_n)$. Since $r_{k+3}(\bar{P}_i) \neq d$, *sd-Eff* implies $\varphi_{nd}(P^{3,1}) = 0$. Then, *ETE* and feasibility imply $\varphi_{1d}(P^{3,1}) = \frac{1}{n-1}$. Hence, by Claim 8, we have $\varphi_{1c}(P^{3,1}) + \varphi_{1d}(P^{3,1}) = \frac{n-2}{n(n-1)} + \frac{1}{n-1} = \frac{2}{n}$. Then, *sd-SP* implies $\varphi_{1d}(P^{3,2}) + \varphi_{1c}(P^{3,2}) = \varphi_{1c}(P^{3,1}) + \varphi_{1d}(P^{3,1}) = \frac{2}{n}$. Last, by *ETE* and feasibility, we have $\varphi_{ic}(P^{3,2}) + \varphi_{id}(P^{3,2}) = \frac{2-\frac{2}{n}-\frac{2}{n}}{n-2} = \frac{2}{n}$ for all $i = 2, \dots, n-1$.

Next, we adopt an induction hypothesis: Given $2 < m \leqslant \bar{n}$ (if n is even), or $2 < m \leqslant \bar{n}+1$ (if n is odd), for all $2 \leqslant l < m$, we have $\varphi_{ic}(P^{3,l}) + \varphi_{id}(P^{3,l}) = \frac{2}{n}$ for all $i \in I$. We show $\varphi_{ic}(P^{3,m}) + \varphi_{id}(P^{3,m}) = \frac{2}{n}$ for all $i \in I$.

First, by *sd-Eff*, we have $\varphi_{nd}(P^{3,m}) = 0$. Next, by Claim 6, *sd-SP* implies $\varphi_{nc}(P^{3,m}) + \varphi_{nb}(P^{3,m}) = \varphi_{nb}(P^{1,m-1}) + \varphi_{nc}(P^{1,m-1}) = \frac{2}{n}$. Since $\varphi_{nb}(P^{3,m}) = 0$ in Step 1, we have $\varphi_{nc}(P^{3,m}) = \frac{2}{n}$, and hence $\varphi_{nc}(P^{3,m}) + \varphi_{nd}(P^{3,m}) = \frac{2}{n}$. Next, by *sd-SP* and the induction hypothesis, we have $\varphi_{m-1\,c}(P^{3,m}) + \varphi_{m-1\,d}(P^{3,m}) = \varphi_{m-1\,c}(P^{3,m-1}) + \varphi_{m-1\,d}(P^{3,m-1}) = \frac{2}{n}$. Then, *ETE* implies $\varphi_{ic}(P^{3,m}) + \varphi_{id}(P^{3,m}) = \frac{2}{n}$ for all $i = 1, \dots, m-1$. Last, by *ETE* and feasibility, we have $\varphi_{jc}(P^{3,m}) + \varphi_{jd}(P^{3,m}) = \frac{2-\frac{2}{n}-(m-1)\frac{2}{n}}{n-m} = \frac{2}{n}$ for all $j = m, \dots, n-1$. This completes the verification of the induction hypothesis, and hence proves the second observation.

Step 3: Now, we show the random assignment over c at each preference profile.

First, by Claim 6, *sd-SP* implies $\varphi_{nc}(P^{3,2}) + \varphi_{nb}(P^{3,2}) = \varphi_{nb}(P^{1,1}) + \varphi_{nc}(P^{1,1}) = \frac{2}{n}$. Since $\varphi_{nb}(P^{3,2}) = 0$ in Step 1, we have $\varphi_{nc}(P^{3,2}) = \frac{2}{n}$. Next, *sd-Eff* implies $\varphi_{1c}(P^{3,2}) = 0$. Then, by *ETE* and feasibility, we have $\varphi_{ic}(P^{3,2}) = \frac{1-\frac{2}{n}}{n-2} = \frac{n-2}{n(n-2)}$ for all $i = 2, \dots, n-1$.

Next, we adopt an induction hypothesis: Given $2 < m \leqslant \bar{n}$ (if n is even), or $2 < m \leqslant \bar{n}+1$ (if n is odd), for all $2 \leqslant l < m$, we have (i) $\varphi_{ic}(P^{3,l}) = 0$ for all $i = 1, \dots, l-1$, (ii) $\varphi_{jc}(P^{3,l}) = \frac{n-2}{n(n-l)}$ for all $j = l, \dots, n-1$, and (iii) $\varphi_{nc}(P^{3,l}) = \frac{2}{n}$. We show $\varphi_{ic}(P^{3,m}) = 0$ for all $i = 1, \dots, m-1$, $\varphi_{jc}(P^{3,m}) = \frac{n-2}{n(n-m)}$ for all $j = m, \dots, n-1$, and $\varphi_{nc}(P^{3,m}) = \frac{2}{n}$.

First, by Claim 6, *sd-SP* implies $\varphi_{nc}(P^{3,m}) + \varphi_{nb}(P^{3,m}) = \varphi_{nb}(P^{1,m-1}) + \varphi_{nc}(P^{1,m-1}) = \frac{2}{n}$. Since $\varphi_{nb}(P^{3,m}) = 0$ in Step 1, we have $\varphi_{nc}(P^{3,m}) = \frac{2}{n}$. Next, suppose that there exists $i^* \in \{1, \dots, m-1\}$ such that $\varphi_{i^*c}(P^{3,m}) > 0$. Since every agent other than $1, \dots, m-1$ prefers c to d, *sd-Eff* implies $\varphi_{jd}(P^{3,m}) = 0$ for all $j = m, \dots, n$. Then, *ETE* and feasibility imply $\varphi_{m-1\,d}(P^{3,m}) = \frac{1}{m-1}$. Hence, $\varphi_{m-1\,d}(P^{3,m}) + \varphi_{m-1\,c}(P^{3,m}) > \frac{1}{m-1}$. If $r_{k+3}(\bar{P}_i) = d$, by Observation 1, we have

$$\frac{4}{n} = \left[\varphi_{m-1\,b}(P^{3,m}) + \varphi_{m-1\,a}(P^{3,m})\right] + \left[\varphi_{m-1\,d}(P^{3,m}) + \varphi_{m-1\,c}(P^{3,m})\right] > \alpha(m) + \frac{1}{m-1}.$$

However, if n is even, one can easily show that

$$\begin{aligned}\left[\alpha(m) + \frac{1}{m-1}\right] - \frac{4}{n} =& \frac{2n^2 - (2m-1)n + 1}{n(n-1)[n-(m-1)]} + \frac{1}{m-1} - \frac{4}{n} \\ \geqslant& \frac{2n^2 - (2m-1)n + 1}{n(n-1)[n-(m-1)]} + \frac{2}{n-2} - \frac{4}{n} \\ =& \frac{5n^2 - 6mn + n + 8m - 10}{n(n-1)(n-2)[n-(m-1)]} \\ \geqslant& \frac{2n^2 + n + 8m - 10}{n(n-1)(n-2)[n-(m-1)]} > 0,\end{aligned}$$

and if n is odd, one can easily show that

$$\begin{aligned}\left[\alpha(m)+\frac{1}{m-1}\right]-\frac{4}{n}=&\frac{2n^2-(2m-1)n+1}{n(n-1)[n-(m-1)]}+\frac{1}{m-1}-\frac{4}{n}\\ \geqslant&\frac{2n^2-(2m-1)n+1}{n(n-1)[n-(m-1)]}+\frac{2}{n-1}-\frac{4}{n}\\ =&\frac{3n+5-4m}{n(n-1)[n-(m-1)]}\\ \geqslant&\frac{n+3}{n(n-1)[n-(m-1)]}>0.\end{aligned}$$

Contradiction! If $r_{k+3}(\bar{P}_i)\neq d$, by Observation 2, we have $\frac{2}{n}=\varphi_{m-1\,d}(P^{3,m})+\varphi_{m-1\,c}(P^{3,m})>\frac{1}{m-1}$. However, one can easily observe $\frac{2}{n}<\frac{1}{m-1}$. Contradiction! Therefore, $\varphi_{ic}(P^{3,m})=0$ for all $i=1,\ldots,m-1$. Last, by *ETE* and feasibility, we have $\varphi_{jc}(P^{3,m})=\frac{1-\frac{2}{n}}{n-m}=\frac{n-2}{n(n-m)}$ for all $j=m,\ldots,n-1$. This completes the verification of the induction hypothesis. We hence finish the specification of the random assignment over c at each preference profile of profile group III under *the double elevating property*. This proves the claim. ■

Claim 10 *In profile group IV, for each $m=1,\ldots,\bar{n}$, at $P^{4,m}=(\hat{P}_1,\ldots,\hat{P}_{n-m},P_{n-m+1},\ldots,P_{n-1},\bar{P}_n)$, the random assignment $\varphi(P^{4,m})$ over a,b and c is specified below.*

	a	b	c
$1,\ldots,n-m$:	$\frac{n-2m}{n(n-m)}$	$\frac{1}{n-m}$	$\frac{1}{n}$
$n-m+1,\ldots,n-1$:	$\frac{2}{n}$	0	$\frac{1}{n}$
n:	$\frac{2}{n}$	0	$\frac{1}{n}$

Under the local elevating property

	a	b	c
$1,\ldots,n-m$:	$\frac{n-2m}{n(n-m)}$	$\frac{1}{n-m}$	$\frac{n-2m}{n(n-m)}$
$n-m+1,\ldots,n-1$:	$\frac{2}{n}$	0	$\frac{2}{n}$
n:	$\frac{2}{n}$	0	$\frac{2}{n}$

Under the double elevating property

Proof: The proof of this claim consists of two steps. In the first step, we specify the random assignment over a and b under both *the local elevating* and the *double elevating properties*. By Claim 5, we then automatically obtain the random assignment over c under *the local elevating property*. In the second step, we specify the random assignment over c under *the double elevating property*.

Step 1: First, by Claim 7, *sd-SP* implies $\varphi_{na}(P^{4,1})=\varphi_{na}(P^{2,1})=\frac{2}{n}$. Next, *sd-Eff* implies $\varphi_{nb}(P^{4,1})=0$. Then, by *ETE* and feasibility, we have $\varphi_{ib}(P^{4,1})=\frac{1}{n-1}$ and $\varphi_{ia}(P^{4,1})=\frac{1-\frac{2}{n}}{n-1}=\frac{n-2}{n(n-1)}$ for all $i=1,\ldots,n-1$.

Next, we provide an induction hypothesis: Given $1<m\leqslant\bar{n}$, for all $2\leqslant l<m$, we have the random assignment $\varphi(P^{4,l})$ over a and b specified below.

	a	b
$1,\ldots,n-l$:	$\frac{n-2l}{n(n-l)}$	$\frac{1}{n-l}$
$n-l+1,\ldots,n-1$:	$\frac{2}{n}$	0
n:	$\frac{2}{n}$	0

We specify the random assignment $\varphi(P^{4,m})$ over a and b to complete the verification of the induction hypothesis.

First, by Claim 7, *sd-SP* implies $\varphi_{na}(P^{4,m}) = \varphi_{na}(P^{2,m}) = \frac{2}{n}$. Next, *sd-Eff* implies $\varphi_{nb}(P^{4,m}) = 0$. By the induction hypothesis, *sd-SP* implies $\varphi_{n-m+1\,a}(P^{4,m}) + \varphi_{n-m+1\,b}(P^{4,m}) = \varphi_{n-m+1\,b}(P^{4,m-1}) + \varphi_{n-m+1\,a}(P^{4,m-1}) = \frac{2}{n}$. Then, *ETE* implies $\varphi_{ja}(P^{4,m}) + \varphi_{jb}(P^{4,m}) = \frac{2}{n}$ for all $j = n-m+1,\ldots,n-1$. Suppose that there exists $j^* \in \{n-m+1,\ldots,n-1\}$ such that $\varphi_{j^*b}(P^{4,m}) > 0$. Since every agent other than $n-m+1,\ldots,n-1$ prefers b to a, *sd-Eff* implies $\varphi_{ia}(P^{4,m}) = 0$ for all $i = 1,\ldots,n-m,n$. Consequently, *ETE* and feasibility imply $\varphi_{j^*a}(P^{4,m}) = \frac{1-\frac{2}{n}}{m-1} = \frac{n-2}{n(m-1)}$. Hence, $\frac{2}{n} = \varphi_{j^*a}(P^{4,m}) + \varphi_{j^*b}(P^{4,m}) > \frac{n-2}{n(m-1)}$. However, one can easily show $\frac{2}{n} \leqslant \frac{n-2}{n(m-1)}$. Therefore, $\varphi_{jb}(P^{4,m}) = 0$ for all $j = n-m+1,\ldots,n-1$. Hence, $\varphi_{ja}(P^{4,m}) = \frac{2}{n}$ for all $j = n-m+1,\ldots,n-1$. Last, by *ETE* and feasibility, we have $\varphi_{ib}(P^{4,m}) = \frac{1}{n-m}$ and $\varphi_{ia}(P^{4,m}) = \frac{1-m\times\frac{2}{n}}{n-m} = \frac{n-2m}{n(n-m)}$ for all $i = 1,\ldots,n-m$. In conclusion, we have the random assignment $\varphi(P^{4,m})$ over a and b specified below.

	a	b
$1,\ldots,n-m$:	$\frac{n-2m}{n(n-m)}$	$\frac{1}{n-m}$
$n-m+1,\ldots,n-1$:	$\frac{2}{n}$	0
n:	$\frac{2}{n}$	0

This completes the verification of the induction hypothesis and Step 1.

Step 2: Under *the local elevating property*, by applying Claim 5, we have the random assignment over c at each preference profile. We focus on specifying the random assignment over c at each preference profile under *the double elevating property*.

First, by Claim 7, *sd-SP* implies $\varphi_{nc}(P^{4,1}) + \varphi_{nb}(P^{4,1}) = \varphi_{nc}(P^{2,1}) + \varphi_{nb}(P^{2,1}) = \frac{2}{n}$. Since $\varphi_{nb}(P^{4,1}) = 0$ in Step 1, we have $\varphi_{nc}(P^{4,1}) = \frac{2}{n}$. Then, by *ETE* and feasibility, we have $\varphi_{ic}(P^{4,1}) = \frac{1-\frac{2}{n}}{n-1} = \frac{n-2}{n(n-1)}$. Meanwhile, note that *sd-Eff* implies $\varphi_{nd}(P^{4,1}) = 0$. Therefore, $\varphi_{id}(P^{4,1}) = \frac{1}{n-1}$ for all $i = 1,\ldots,n-1$ by *ETE* and feasibility. Hence, $\varphi_{ic}(P^{4,1}) + \varphi_{id}(P^{4,1}) = \frac{2}{n}$ for all $i \in I$.

Next, we adopt an induction hypothesis: Given $1 < m \leqslant \bar{n}$, for all $1 \leqslant l < m$, we have

(i) $\varphi_{ic}(P^{4,l}) = \frac{n-2l}{n(n-l)}$ for all $i = 1,\ldots,n-l$,
$\varphi_{jc}(P^{4,l}) = \frac{2}{n}$ for all $j = n-l+1,\ldots,n-1$, and
$\varphi_{nc}(P^{4,l}) = \frac{2}{n}$, and

(ii) $\varphi_{ic}(P^{4,l}) + \varphi_{id}(P^{4,l}) = \frac{2}{n}$ for all $i \in I$.

We show that (i) $\varphi_{ic}(P^{4,m}) = \frac{n-2m}{n(n-m)}$ for all $i = 1,\ldots,n-m$, $\varphi_{jc}(P^{4,m}) = \frac{2}{n}$ for all $j = n-m+1,\ldots,n-1$, and $\varphi_{nc}(P^{4,m}) = \frac{2}{n}$, and (ii) $\varphi_{ic}(P^{4,m}) + \varphi_{id}(P^{4,m}) = \frac{2}{n}$ for all $i \in I$.

First, by Claim 7, *sd-SP* implies $\varphi_{nc}(P^{4,m}) + \varphi_{nb}(P^{4,m}) = \varphi_{nc}(P^{2,m}) + \varphi_{nb}(P^{2,m}) = \frac{2}{n}$. Since $\varphi_{nb}(P^{4,m}) = 0$ in Step 1, we have $\varphi_{nc}(P^{4,m}) = \frac{2}{n}$. Note that $\varphi_{nd}(P^{2,m}) = 0$ by Claim 7, P_i and $\bar{P}_i$ have the same set of top $k+2$ ranked objects, $d = r_{k+3}(P_i)$ and $d = r_\nu(\bar{P}_i)$ for some $\nu \geqslant k+3$. Therefore, *sd-SP* implies $0 = \varphi_{nd}(P^{2,m}) \geqslant \varphi_{nd}(P^{4,m})$. Hence, $\varphi_{nd}(P^{4,m}) = 0$. Thus, we have $\varphi_{nc}(P^{4,m}) + \varphi_{nd}(P^{4,m}) = \frac{2}{n}$.

Next, by *sd-SP* and the induction hypothesis, we have $\varphi_{n-m+1\,c}(P^{4,m}) + \varphi_{n-m+1\,d}(P^{4,m}) = \varphi_{n-m+1\,d}(P^{4,m-1}) + \varphi_{n-m+1\,c}(P^{4,m-1}) = \frac{2}{n}$. Then, *ETE* implies $\varphi_{jc}(P^{4,m}) + \varphi_{jd}(P^{4,m}) = \frac{2}{n}$ for all $j = n-m+1,\ldots,n-1$. Then, by *ETE* and feasibility, we have $\varphi_{ic}(P^{4,m}) + \varphi_{id}(P^{4,m}) = \frac{2-(m-1)\frac{2}{n}-\frac{2}{n}}{n-m} = \frac{2}{n}$ for all $i = 1,\ldots,n-m$. Hence, $\varphi_{ic}(P^{4,m}) + \varphi_{id}(P^{4,m}) = \frac{2}{n}$ for all $i \in I$.

Next, suppose that there exists $j^* \in \{n-m+1, \dots, n-1\}$ such that $\varphi_{j^*d}(P^{4,m}) > 0$. Since every agent other than $n-m+1, \dots, n-1$ prefers d to c, *sd-Eff* implies $\varphi_{ic}(P^{4,m}) = 0$ for all $i = 1, \dots, n-m, n$. Then, *ETE* and feasibility imply $\varphi_{j^*c}(P^{4,m}) = \frac{1-\frac{2}{n}}{m-1} = \frac{n-2}{n(m-1)}$. Thus, $\frac{2}{n} = \varphi_{j^*c}(P^{4,m}) + \varphi_{j^*d}(P^{4,m}) > \frac{n-2}{n(m-1)}$. However, one can easily show $\frac{2}{n} \leqslant \frac{n-2}{n(m-1)}$. Contradiction! Therefore, $\varphi_{jd}(P^{4,m}) = 0$ for all $j = n-m+1, \dots, n-1$. Hence, $\varphi_{jc}(P^{4,m}) = \frac{2}{n}$ for all $j = n-m+1, \dots, n-1$.

Last, by *ETE* and feasibility, we have $\varphi_{ic}(P^{4,m}) = \frac{1-(m-1)\frac{2}{n}-\frac{2}{n}}{n-m} = \frac{n-2m}{n(n-m)}$ for all $i = 1, \dots, n-m$. This completes the verification of the induction hypothesis. We hence finish the specification of the random assignment over c at each preference profile under *the double elevating property*. This proves the claim. ■

Now, we induce the contradiction for the case of an even number of agents. Let $n \geqslant 4$ be an even integer. Thus, $\bar{n} = \frac{n}{2}$. Notice that $P^{3,\bar{n}}$ and $P^{4,\bar{n}}$ differ exactly in agent $\bar{n}$'s preference, i.e., $P^{3,\bar{n}}_{\bar{n}} = P_i$ and $P^{4,\bar{n}}_{\bar{n}} = \hat{P}_i$ in Definition 3 or 4. Then, *sd-SP* implies $\varphi_{\bar{n}a}(P^{3,\bar{n}}) + \varphi_{\bar{n}b}(P^{3,\bar{n}}) = \varphi_{\bar{n}a}(P^{4,\bar{n}}) + \varphi_{\bar{n}b}(P^{4,\bar{n}})$. Thus, by Claims 9 and 10, we have

$$\begin{aligned} 0 &= \left[\varphi_{\bar{n}a}(P^{3,\bar{n}}) + \varphi_{\bar{n}b}(P^{3,\bar{n}})\right] - \left[\varphi_{\bar{n}a}(P^{4,\bar{n}}) + \varphi_{\bar{n}b}(P^{4,\bar{n}})\right] \\ &= \left[\frac{1}{[n-(\frac{n}{2}-1)]} + \frac{1-(\frac{n}{2}-1)\alpha(\frac{n}{2})}{n-\frac{n}{2}}\right] - \frac{2}{n} = \frac{2}{n^2(n-1)}. \text{ Contradiction!} \end{aligned}$$

In conclusion, in the case of an even number of agents, domain $\mathbb{D}$ satisfying *the local elevating property* or *the double elevating property* admits no *sd-SP*, *sd-Eff* and *ETE* rule.

Before turning to the case of an odd number of agents, we make one note on the case of an even number of agents. In all Claims 6 - 10, *only* the specification of the random assignment over a and b (correspondingly, Step 1 in the proof of each claim) is used to establish the impossibility result. The specification of the random assignment over c is established for the following-up investigation in the case of an odd number of agents.

Now, we consider the case of an odd number of agents. Henceforth, let $n \geqslant 5$ be an odd integer. Thus, $\bar{n} = \frac{n-1}{2}$. Claim 11 below is a preparation which will be used in establishing the following-up claims.

Claim 11 *In profile group III, under both the local elevating and double elevating properties, for each $m = 2, \dots, \bar{n}, \bar{n}+1$, at $P^{3,m} = (\hat{P}_1, \dots, \hat{P}_{m-1}, P_m, \dots, P_{n-1}, \bar{P}_n)$, we have $\varphi_{n-1\,b}(P^{3,m}) + \varphi_{n-1\,c}(P^{3,m}) < \frac{2}{n-1}$.*

Proof: We first consider the situation under *the local elevating property*. Fix $2 \leqslant m \leqslant \bar{n}+1$. By Claim 9, we know $\varphi_{n-1\,b}(P^{3,m}) + \varphi_{n-1\,c}(P^{3,m}) = \frac{3}{n} - \frac{1}{n-(m-1)}$. Then, it is easy to show that $\frac{3}{n} - \frac{1}{n-(m-1)} - \frac{2}{n-1} \leqslant \frac{3}{n} - \frac{1}{n-1} - \frac{2}{n-1} < 0$.

Next, we consider the situation under *the double elevating property*. By Claim 9, for each $m = 2, \dots, \bar{n}, \bar{n}+1$, we have

$$\begin{aligned} \frac{2}{n-1} - \left[\varphi_{n-1\,b}(P^{3,m}) + \varphi_{n-1\,c}(P^{3,m})\right] &= \frac{2}{n-1} - \left[\frac{1-(m-1)\times\alpha(m)}{n-m} + \frac{n-2}{n(n-m)}\right] \\ &= \frac{2}{n-1} - \frac{2(n-1)^2[n-(m-1)] - (m-1)[2n^2-(2m-1)n+1]}{n(n-1)(n-m)[n-(m-1)]} \\ &= \frac{2n-3m+3}{n(n-m)[n-(m-1)]} \geqslant \frac{n+3}{2n(n-m)[n-(m-1)]} > 0. \end{aligned}$$

This proves the claim. ■

Claim 12 *In profile group V, under both the local elevating and double elevating properties, at* $P^{5,1} = (P_1, \ldots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$*, the random assignment* $\varphi(P^{5,1})$ *over* a, b *and* c *is specified below.*

$$\begin{array}{rccc} & a & b & c \\ 1, \ldots, n-2: & \frac{1}{n} & \frac{1}{n-2} & \frac{n-4}{n(n-2)} \\ n-1, n: & \frac{1}{n} & 0 & \frac{2}{n} \end{array}$$

Proof: First, by Claim 8, *sd-SP* implies $\varphi_{n-1\,a}(P^{5,1}) = \varphi_{n-1\,a}(P^{3,1}) = \frac{1}{n}$ and $\varphi_{n-1\,c}(P^{5,1}) + \varphi_{n-1\,b}(P^{5,1}) = \varphi_{n-1\,b}(P^{3,1}) + \varphi_{n-1\,c}(P^{3,1}) = \frac{2}{n}$. Next, suppose $\varphi_{n-1\,b}(P^{5,1}) > 0$. Since every agent other than $n-1$ and n prefers b to c, *sd-Eff* implies $\varphi_{ic}(P^{5,1}) = 0$ for all $i = 1, \ldots, n-2$. Consequently, *ETE* and feasibility imply $\varphi_{n-1\,c}(P^{5,1}) = \frac{1}{2}$. Thus, we have $\frac{2}{n} = \varphi_{n-1\,c}(P^{5,1}) + \varphi_{n-1\,b}(P^{5,1}) > \frac{1}{2}$. Contradiction! Therefore, $\varphi_{n-1\,b}(P^{5,1}) = 0$, and hence $\varphi_{n-1\,c}(P^{5,1}) = \frac{2}{n}$. Moreover, *ETE* implies $\varphi_{na}(P^{5,1}) = \frac{1}{n}$, $\varphi_{nb}(P^{5,1}) = 0$ and $\varphi_{nc}(P^{5,1}) = \frac{2}{n}$. Last, by *ETE* and feasibility, we have $\varphi_{ia}(P^{5,1}) = \frac{1-2\times\frac{1}{n}}{n-2} = \frac{1}{n}$, $\varphi_{ib}(P^{5,1}) = \frac{1}{n-2}$ and $\varphi_{ic}(P^{5,1}) = \frac{1-2\times\frac{2}{n}}{n-2} = \frac{n-4}{n(n-2)}$ for all $i = 1, \ldots, n-2$. ■

Claim 13 *In profile group V, under both the local elevating and double elevating properties, for each* $m = 2, \ldots, \bar{n}, \bar{n}+1$*, at* $P^{5,m} = (\hat{P}_1, \ldots, \hat{P}_{m-1}, P_m, \ldots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$*, the random assignment* $\varphi(P^{5,m})$ *over* a *and* b *is specified below*

$$\begin{array}{rcc} & a & b \\ 1, \ldots, m-1: & 0 & \gamma(m) \\ m, \ldots, n-2: & \frac{1}{n-(m-1)} & \frac{1-(m-1)\gamma(m)}{n-(m+1)} \\ n-1, n: & \frac{1}{n-(m-1)} & 0 \end{array}$$

where $\gamma(m) = \frac{2n^4 - 2(2m+1)n^3 + 2(m+1)^2n^2 - 2(m^2+m+1)n + 4}{n(n-1)(n-2)[n-(m-1)](n-m)}$.

Proof: The proof of this claim consists of three steps. In the first step, we specify the random assignment over a and b for agents $n-1$ and n. In the second step, we show that equation $\gamma(m)$ is decreasing from $m = 2$ to $m = \bar{n}+1$, while the third step specifies the random assignment over a and b for agents $1, \ldots, m-1$ and $m, \ldots, n-2$.

Step 1: Given $2 \leqslant m \leqslant \bar{n}+1$, by Claim 9, *sd-SP* implies $\varphi_{n-1\,a}(P^{5,m}) = \varphi_{n-1\,a}(P^{3,m}) = \frac{1}{n-(m-1)}$ and $\varphi_{n-1\,c}(P^{5,m}) + \varphi_{n-1\,b}(P^{5,m}) = \varphi_{n-1\,b}(P^{3,m}) + \varphi_{n-1\,c}(P^{3,m})$. By Claim 11, we know $\varphi_{n-1\,c}(P^{5,m}) + \varphi_{n-1\,b}(P^{5,m}) < \frac{2}{n-1}$. Suppose $\varphi_{n-1\,b}(P^{5,m}) > 0$. Since every agent other than $n-1$ and n prefers b to c, *sd-Eff* implies $\varphi_{ic}(P^{5,m}) = 0$ for all $i = 1, \ldots, n-2$. Then, *ETE* and feasibility imply $\varphi_{n-1\,c}(P^{5,m}) = \frac{1}{2}$. Consequently, we have $\frac{2}{n-1} > \varphi_{n-1\,c}(P^{5,m}) + \varphi_{n-1\,b}(P^{5,m}) > \frac{1}{2}$. Contradiction! Therefore, $\varphi_{n-1\,b}(P^{5,m}) = 0$. Then, by *ETE*, we have $\varphi_{na}(P^{5,m}) = \frac{1}{n-(m-1)}$ and $\varphi_{nb}(P^{5,m}) = 0$. In conclusion, for each $m = 2, \ldots, \bar{n}, \bar{n}+1$, the random assignment $\varphi(P^{5,m})$ over a and b for agents $n-1$ and n is specified below.

$$\begin{array}{rcc} & a & b \\ n-1, n: & \frac{1}{n-(m-1)} & 0 \end{array}$$

Step 2: Given $3 \leqslant m \leqslant \bar{n}+1$, we have

$$\begin{aligned}\gamma(m)-\gamma(m-1) =& \frac{2n^4-2(2m+1)n^3+2(m+1)^2n^2-2(m^2+m+1)n+4}{n(n-1)(n-2)[n-(m-1)](n-m)} \\ & -\frac{2n^4-2(2m-1)n^3+2m^2n^2-2(m^2-m+1)n+4}{n(n-1)(n-2)[n-(m-2)][n-(m-1)]} \\ =& \frac{-2(n^2-mn+2)}{n(n-1)[n-(m-2)][n-(m-1)](n-m)} < 0.\end{aligned}$$

Therefore, $\gamma(m)$ is decreasing from $m=2$ to $m=\bar{n}+1$.

Step 3: For each $m=2,\dots,\bar{n},\bar{n}+1$, we specify the random assignment over a and b for agents $1,\dots,m-1$ and $m,\dots,n-2$.

First, by Claim 12, *sd-SP* implies $\varphi_{1b}(P^{5,2})+\varphi_{1a}(P^{5,2})=\varphi_{1b}(P^{5,1})+\varphi_{1a}(P^{5,1})=\frac{1}{n}+\frac{1}{n-2}=\gamma(2)$. Next, since *sd-Eff* implies $\varphi_{1a}(P^{5,2})=0$, we have $\varphi_{1b}(P^{5,2})=\gamma(2)$. Then, by *ETE* and feasibility, we have $\varphi_{ia}(P^{5,2})=\frac{1-2\times\frac{1}{n-(2-1)}}{n-3}=\frac{1}{n-(2-1)}$ and $\varphi_{ib}(P^{5,2})=\frac{1-(2-1)\times\gamma(2)}{n-(2+1)}$ for all $i=2,\dots,n-2$.

Next, we adopt an induction hypothesis: Given $2<m\leqslant\bar{n}+1$, for all $2\leqslant l<m$, the random assignments $\varphi(P^{5,l})$ over a and b for all agents $1,\dots,l-1$ and $l,\dots,n-2$ are specified as follows:

	a	b
$1,\dots,l-1:$	0	$\gamma(l)$
$l,\dots,n-2:$	$\frac{1}{n-(l-1)}$	$\frac{1-(l-1)\gamma(l)}{n-(l+1)}$

We specify the random assignment $\varphi(P^{5,m})$ over a and b for all agents $1,\dots,m-1$ and $m,\dots,n-2$ to complete the verification of the induction hypothesis. First, by the induction hypothesis, *sd-SP* implies

$$\begin{aligned}\varphi_{m-1\,b}(P^{5,m})+\varphi_{m-1\,a}(P^{5,m}) =& \varphi_{m-1\,a}(P^{5,m-1})+\varphi_{m-1\,b}(P^{5,m-1}) \\ =& \frac{1}{n-(m-2)}+\frac{1-(m-2)\times\gamma(m-1)}{n-m} \\ =& \frac{1}{n-(m-2)}+\frac{1-(m-2)\times\left[\frac{2n^4-2(2m-1)n^3+2m^2n^2-2(m^2-m+1)n+4}{n(n-1)(n-2)[n-(m-2)][n-(m-1)]}\right]}{n-m} \\ =& \frac{2n^4-2(2m+1)n^3+2(m+1)^2n^2-2(m^2+m+1)n+4}{n(n-1)(n-2)[n-(m-1)](n-m)} \\ =& \gamma(m).\end{aligned}$$

Furthermore, *ETE* implies $\varphi_{ib}(P^{5,m})+\varphi_{ia}(P^{5,m})=\gamma(m)$ for all $i=1,\dots,m-1$.

Next, suppose that there exists $i^*\in\{1,\dots,m-1\}$ such that $\varphi_{i^*a}(P^{5,m})>0$. Since every agent other than $1,\dots,m-1$ prefers a to b, *sd-Eff* implies $\varphi_{jb}(P^{5,m})=0$ for all $j=m,\dots,n$. Consequently, *ETE* and feasibility imply $\varphi_{i^*b}(P^{5,m})=\frac{1}{m-1}$. Hence, $\gamma(m)=\varphi_{i^*b}(P^{5,m})+\varphi_{i^*a}(P^{5,m})>\frac{1}{m-1}$. Since $m>2$, by Step 2, we have

$$\gamma(m)\leqslant\gamma(3)=\frac{2n^4-14n^3+32n^2-26n+4}{n(n-1)(n-2)(n-2)(n-3)}=\frac{2}{n-1}-\frac{2}{n(n-1)(n-2)(n-3)}<\frac{1}{m-1}.$$

Contradiction! Therefore, $\varphi_{ia}(P^{5,m}) = 0$ for all $i = 1, \ldots, m-1$. Hence, $\varphi_{ib}(P^{5,m}) = \gamma(m)$ for all $i = 1, \ldots, m-1$.

Last, by *ETE* and feasibility, we have $\varphi_{ia}(P^{5,m}) = \frac{1-2\times\frac{1}{n-(m-1)}}{n-(m+1)} = \frac{1}{n-(m-1)}$ and $\varphi_{ib}(P^{5,m}) = \frac{1-(m-1)\gamma(m)}{n-(m+1)}$ for all $i = m, \ldots, n-2$. In conclusion, the random assignment $\varphi(P^{5,m})$ over a and b for all agents $1, \ldots, m-1$ and $m, \ldots, n-2$ is specified below.

$$\begin{array}{rcc} & a & b \\ 1,\ldots,m-1: & 0 & \gamma(m) \\ m,\ldots,n-2: & \frac{1}{n-(m-1)} & \frac{1-(m-1)\times\gamma(m)}{n-(m+1)} \end{array}$$

This completes the verification of the induction hypothesis, and hence proves the claim. ■

Claim 14 *Under both the local elevating and double elevating properties, for each $m = 2, \ldots, \bar{n}$, at $P^{6,m} = (\hat{P}_1, \ldots, \hat{P}_{n-m}, P_{n-m+1}, \ldots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$, we have $\varphi_{ia}(P^{6,m}) + \varphi_{ib}(P^{6,m}) = \frac{2}{n}$ for all $i \in I$.*

Proof: We first consider profile $P^{6,2} = (\hat{P}_1, \ldots, \hat{P}_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$, and show $\varphi_{ia}(P^{6,2})+\varphi_{ib}(P^{6,2}) = \frac{2}{n}$ for all $i \in I$. By Claim 10, *sd-SP* implies (i) $\varphi_{n-1\,a}(P^{6,2}) = \varphi_{n-1\,a}(P^{4,2}) = \frac{2}{n}$, (ii) $\varphi_{n-1\,c}(P^{6,2})+\varphi_{n-1\,b}(P^{6,2}) = \varphi_{n-1\,b}(P^{4,2}) + \varphi_{n-1\,c}(P^{4,2}) = \frac{1}{n}$ under *the local elevating property*, and (iii) $\varphi_{n-1\,c}(P^{6,2}) + \varphi_{n-1\,b}(P^{6,2}) = \varphi_{n-1\,b}(P^{4,2}) + \varphi_{n-1\,c}(P^{4,2}) = \frac{2}{n}$ under *the double elevating property*. Next, suppose $\varphi_{n-1\,b}(P^{6,2}) > 0$. Since every agent other than $n-1$ and n prefers b to c, *sd-Eff* implies $\varphi_{ic}(P^{6,2}) = 0$ for all $i = 1, \ldots, n-2$. Then, *ETE* and feasibility imply $\varphi_{n-1\,c}(P^{6,2}) = \frac{1}{2}$. Consequently, we have $\frac{1}{n} = \varphi_{n-1\,c}(P^{6,2})+\varphi_{n-1\,b}(P^{6,2}) > \frac{1}{2}$ under *the local elevating property*, and $\frac{2}{n} = \varphi_{n-1\,c}(P^{6,2}) + \varphi_{n-1\,b}(P^{6,2}) > \frac{1}{2}$ under *the double elevating property*. Contradiction! Therefore, $\varphi_{n-1\,b}(P^{6,2}) = 0$. Thus, $\varphi_{n-1\,a}(P^{6,2}) + \varphi_{n-1\,b}(P^{6,2}) = \frac{2}{n}$ and $\varphi_{na}(P^{6,2}) + \varphi_{nb}(P^{6,2}) = \frac{2}{n}$ by *ETE*. Last, by *ETE* and feasibility, we have $\varphi_{ia}(P^{6,2}) + \varphi_{ib}(P^{6,2}) = \frac{2-2\times\frac{2}{n}}{n-2} = \frac{2}{n}$ for all $i = 1, \ldots, n-2$. In conclusion, $\varphi_{ia}(P^{6,2}) + \varphi_{ib}(P^{6,2}) = \frac{2}{n}$ for all $i \in I$.

Next, we adopt an induction hypothesis: Given $2 < m \leqslant \bar{n}$, for all $2 \leqslant l < m$, $\varphi_{ia}(P^{6,l}) + \varphi_{ib}(P^{6,l}) = \frac{2}{n}$ for all $i \in I$. We show $\varphi_{ia}(P^{6,m}) + \varphi_{ib}(P^{6,m}) = \frac{2}{n}$ for all $i \in I$.

By Claim 10, *sd-SP* implies (i) $\varphi_{n-1\,a}(P^{6,m}) = \varphi_{n-1\,a}(P^{4,m}) = \frac{2}{n}$, (ii) $\varphi_{n-1\,c}(P^{6,m}) + \varphi_{n-1\,b}(P^{6,m}) = \varphi_{n-1\,b}(P^{4,m}) + \varphi_{n-1\,c}(P^{4,m}) = \frac{1}{n}$ under *the local elevating property*, and (iii) $\varphi_{n-1\,c}(P^{6,m}) + \varphi_{n-1\,b}(P^{6,m}) = \varphi_{n-1\,b}(P^{4,m}) + \varphi_{n-1\,c}(P^{4,m}) = \frac{2}{n}$ under *the double elevating property*. Next, suppose $\varphi_{n-1\,b}(P^{6,m}) > 0$. Since every agent other than $n-1$ and n prefers b to c, *sd-Eff* implies $\varphi_{ic}(P^{6,m}) = 0$ for all $i = 1, \ldots, n-2$. Then, *ETE* and feasibility imply $\varphi_{n-1\,c}(P^{6,m}) = \frac{1}{2}$. Consequently, we have $\frac{1}{n} = \varphi_{n-1\,c}(P^{6,m}) + \varphi_{n-1\,b}(P^{6,m}) > \frac{1}{2}$ under *the local elevating property*, and $\frac{2}{n} = \varphi_{n-1\,c}(P^{6,m}) + \varphi_{n-1\,b}(P^{6,m}) > \frac{1}{2}$ under *the double elevating property*. Contradiction! Therefore, $\varphi_{n-1\,b}(P^{6,m}) = 0$. Thus, $\varphi_{n-1\,a}(P^{6,m}) + \varphi_{n-1\,b}(P^{6,m}) = \frac{2}{n}$ and $\varphi_{na}(P^{6,m}) + \varphi_{nb}(P^{6,m}) = \frac{2}{n}$ by *ETE*.

Next, by *sd-SP* and the induction hypothesis, we have $\varphi_{n-m+1\,a}(P^{6,m}) + \varphi_{n-m+1\,b}(P^{6,m}) = \varphi_{n-m+1\,a}(P^{6,m-1}) + \varphi_{n-m+1\,b}(P^{6,m-1}) = \frac{2}{n}$. Then, *ETE* implies $\varphi_{ja}(P^{6,m}) + \varphi_{jb}(P^{6,m}) = \frac{2}{n}$ for all $j = n-m+1, \ldots, n-2$. Last, by *ETE* and feasibility, we have $\varphi_{ia}(P^{6,m}) + \varphi_{ib}(P^{6,m}) = \frac{2-2\times\frac{2}{n}-(m-2)\times\frac{2}{n}}{n-m} = \frac{2}{n}$ for all $i = 1, \ldots, n-m$. This completes the verification of the induction hypothesis, and hence proves the claim. ■

Now, we induce the contradiction for the case of an odd number of agents. Let $n \geqslant 5$ be an odd integer. Thus, $\bar{n} = \frac{n-1}{2}$. Notice that $P^{5,\bar{n}+1}$ and $P^{6,\bar{n}}$ differ exactly in preferences of agent $\bar{n}+1$e, i.e., $P^{5,\bar{n}+1}_{\bar{n}+1} = P_i$ and $P^{6,\bar{n}}_{\bar{n}+1} = \hat{P}_i$ in Definition 3 or 4. Then, *sd-SP* implies $\varphi_{\bar{n}+1\,a}(P^{5,\bar{n}+1}) + \varphi_{\bar{n}+1\,b}(P^{5,\bar{n}+1}) = \varphi_{\bar{n}+1\,a}(P^{6,\bar{n}}) + \varphi_{\bar{n}+1\,b}(P^{6,\bar{n}})$. Thus, by Claims 13 and 14, we have

$$\begin{aligned}0 &= \left[\varphi_{\bar{n}+1\,a}(P^{5,\bar{n}+1}) + \varphi_{\bar{n}+1\,b}(P^{5,\bar{n}+1})\right] - \left[\varphi_{\bar{n}+1\,a}(P^{6,\bar{n}}) + \varphi_{\bar{n}+1\,b}(P^{6,\bar{n}})\right]\\ &= \left[\frac{1}{n-\bar{n}} + \frac{1-\bar{n}\times\gamma(\bar{n}+1)}{n-(\bar{n}+2)}\right] - \frac{2}{n} = \frac{-4}{(n-3)(n-2)(n-1)n}.\ \text{Contradiction!}\end{aligned}$$

Therefore, in the case of an odd number of agents, domain $\mathbb{D}$ satisfying *the local elevating property* or *the double elevating property* admits no *sd-SP*, *sd-Eff* and *ETE* rule. This completes the proof of Lemma 1.

C Proof of Lemma 2

Let $\mathbb{D}$ be a *weakly connected* domain violating both *the local elevating* and *double elevating properties*.

Claim 15 *Given distinct $P_i, P_i', P_i'' \in \mathbb{D}$, if $P_i \approx P_i'$ and $P_i' \approx P_i''$, then $P_i \approx P_i''$.*

To prove Claim 15, it suffices to show the following two symmetric statements:

(i) Given $a, b \in A$, if $r_k(P_i) = r_{k+1}(P_i') = b$ and $r_{k+1}(P_i) = r_k(P_i') = a$ for some $1 \leqslant k < n$, then we have either $r_k(P_i'') = a$, $r_{k+1}(P_i'') = b$ and $B(P_i, b) = B(P_i', a) = B(P_i'', a)$, or $r_k(P_i'') = b$, $r_{k+1}(P_i'') = a$ and $B(P_i, b) = B(P_i', a) = B(P_i'', b)$.

(ii) Given $a, b \in A$, if $r_k(P_i') = r_{k+1}(P_i'') = b$ and $r_{k+1}(P_i') = r_k(P_i'') = a$ for some $1 \leqslant k < n$, then we have either $r_k(P_i) = a$, $r_{k+1}(P_i) = b$ and $B(P_i', b) = B(P_i'', a) = B(P_i, a)$, or $r_k(P_i) = b$, $r_{k+1}(P_i) = a$ and $B(P_i', b) = B(P_i'', a) = B(P_i, b)$.

Since both statements are symmetric, we focus on showing statement (i). Since $P_i \approx P_i'$, we can identify several pair(s) of objects $\{(b_l, a_l) : l = 1, \ldots, t\}$ that are locally switched across P_i and P_i', i.e., there exist $1 \leqslant k_1 < k_2 < \cdots < k_t < n$ such that $b_l = r_{k_l}(P_i) = r_{k_l+1}(P_i')$, $a_l = r_{k_l+1}(P_i) = r_{k_l}(P_i')$, $l = 1, \ldots, t$, and $[x\ P_i\ y] \Leftrightarrow [x\ P_i'\ y]$ for all $(x, y) \notin \{(b_l, a_l) : l = 1, \ldots, t\}$. We first show that statement (i) holds for the pair (b_1, a_1).

Since $a_1 = r_{k_1}(P_i')$ and $b_1 = r_{k_1+1}(P_i')$, $P_i' \approx P_i''$ implies $a_1 \in \{r_{k_1-1}(P_i''), r_{k_1}(P_i''), r_{k_1+1}(P_i'')\}$ and $b_1 \in \{r_{k_1}(P_i''), r_{k_1+1}(P_i''), r_{k_1+2}(P_i'')\}$. First, suppose $a_1 \notin \{r_{k_1}(P_i''), r_{k_1+1}(P_i'')\}$. Thus, $a_1 = r_{k_1-1}(P_i'')$. Then, by $P_i' \approx P_i''$, we know (i) $r_{k_1}(P_i'') = r_{k_1-1}(P_i') \equiv x$, (ii) $x\ P_i'!\ a_1$ and $a_1\ P_i''!\ x$, and (iii) $x \neq b_1$. Note that (b_1, a_1) is the first pair (the highest ranked pair) which is locally switched across P_i and P_i'. Then, $P_i \approx P_i'$ implies $r_{k_1-1}(P_i) = r_{k_1-1}(P_i') = x$. We next assert $B(P_i, x) = B(P_i', x) = B(P_i'', a_1)$. The first equality holds evidently. Suppose $B(P_i', x) \neq B(P_i'', a_1)$. Since $|B(P_i', x)| = |B(P_i'', a_1)| = k_1 - 2$, $B(P_i', x) \neq B(P_i'', a_1)$ implies that there exists $y \in B(P_i'', a_1) \backslash B(P_i', x)$. Thus, $y \neq a_1$, $y\ P_i''\ a_1$ and $x\ P_i'\ y$. Since $a_1\ P_i''\ x$, we have $y\ P_i''\ x$. Then, $P_i' \approx P_i''$ implies $x\ P_i'!\ y$ and $y\ P_i''!\ x$, which contradict the fact $x\ P_i'!\ a_1$ and $a_1\ P_i''!\ x$. Therefore, $B(P_i, x) = B(P_i', x) = B(P_i'', a_1)$. Furthermore, since $b_1 \in \{r_{k_1}(P_i''), r_{k_1+1}(P_i''), r_{k_1+2}(P_i'')\}$ and $r_{k_1}(P_i'') = x \neq b_1$, we have two cases: (1) $r_{k_1+1}(P_i'') = b_1$ and (2) $r_{k_1+2}(P_i'') = b_1$. In case (1), we have an instance of *the local elevating property* specified in Table 4 below.

		k_1-1	k_1	k_1+1	
P_i:	$\underbrace{\cdots\cdots}_{B(P_i,x)=B(P'_i,x)}$	$\succ x$	$\succ b_1$	$\succ a_1$	$\succ \cdots$
P'_i:	$\overbrace{\underbrace{\cdots\cdots}_{B(P'_i,x)=B(P''_i,a_1)}}$	$\succ x$	$\succ a_1$	$\succ b_1$	$\succ \cdots$
P''_i:	$\overbrace{\cdots\cdots}$	$\succ a_1$	$\succ x$	$\succ b_1$	$\succ \cdots$

Table 4: An instance of *the local elevating property*

In case (2), by $P'_i \approx P''_i$, we know $r_{k_1+2}(P''_i) = r_{k_1+1}(P'_i) = b_1$ and $r_{k_1+1}(P''_i) = r_{k_1+2}(P'_i) \equiv z$. Consequently, we have an instance of *the double elevating property* specified in Table 5 below.

		k_1-1	k_1	k_1+1	k_1+2	
P_i:	$\underbrace{\cdots\cdots}_{B(P_i,x)=B(P'_i,x)}$	$\succ x$	$\succ b_1$	$\succ a_1$	$\succ \cdot$	$\succ \cdots$
P'_i:	$\overbrace{\underbrace{\cdots\cdots}_{B(P'_i,x)=B(P''_i,a_1)}}$	$\succ x$	$\succ a_1$	$\succ b_1$	$\succ z$	$\succ \cdots$
P''_i:	$\overbrace{\cdots\cdots}$	$\succ a_1$	$\succ x$	$\succ z$	$\succ b_1$	$\succ \cdots$

Table 5: An instance of *the double elevating property*

Hence, in each case, we induce a contradiction. Therefore, $a_1 \in \{r_{k_1}(P''_i), r_{k_1+1}(P''_i)\}$.

Next, suppose $b_1 \notin \{r_{k_1}(P''_i), r_{k_1+1}(P''_i)\}$. Thus, $b_1 = r_{k_1+2}(P''_i)$. Then, by $P'_i \approx P''_i$, we know $r_{k_1+1}(P''_i) = r_{k_1+2}(P'_i) \equiv x$, and hence $x \neq a_1$. Furthermore, since $a_1 \in \{r_{k_1}(P''_i), r_{k_1+1}(P''_i)\}$, we have $r_{k_1}(P''_i) = a_1$. Thus, $a_1 = r_{k_1}(P'_i) = r_{k_1}(P''_i)$. We next assert $B(P_i, b_1) = B(P'_i, a_1) = B(P''_i, a_1)$. The first equality holds evidently. Suppose $B(P'_i, a_1) \neq B(P''_i, a_1)$. Since $|B(P'_i, a_1)| = |B(P''_i, a_1)| = k_1 - 1$, $B(P'_i, a_1) \neq B(P''_i, a_1)$ implies that there exists $y \in B(P''_i, a_1) \backslash B(P'_i, a_1)$. Thus, $y\ P''_i\ a_1$ and $a_1\ P'_i\ y$ which by $P'_i \approx P''_i$ imply $a_1\ P'_i!\ y$ and $y\ P''_i!\ a_1$. This contradicts the fact $a_1 = r_{k_1}(P'_i) = r_{k_1}(P''_i)$. Therefore, $B(P_i, b_1) = B(P'_i, a_1) = B(P''_i, a_1)$. Furthermore, since $P_i \approx P'_i$ and $r_{k_1+1}(P_i) = a_1 \neq x$, we have two cases: (1) $r_{k_1+2}(P_i) = x$ and (2) $r_{k_1+3}(P_i) = x$. In case (1), we have an instance of *the local elevating property* specified in Table 6 below.

		k_1	k_1+1	k_1+2	
P''_i:	$\underbrace{\cdots\cdots}_{B(P''_i,a_1)=B(P'_i,a_1)}$	$\succ a_1$	$\succ x$	$\succ b_1$	$\succ \cdots$
P'_i:	$\overbrace{\underbrace{\cdots\cdots}_{B(P'_i,a_1)=B(P_i,b_1)}}$	$\succ a_1$	$\succ b_1$	$\succ x$	$\succ \cdots$
P_i:	$\overbrace{\cdots\cdots}$	$\succ b_1$	$\succ a_1$	$\succ x$	$\succ \cdots$

Table 6: An instance of *the local elevating property*

In case (2), since $P_i \approx P'_i$, we know $r_{k_1+2}(P'_i) = r_{k_1+3}(P_i) = x$ and $r_{k_1+3}(P'_i) = r_{k_1+2}(P_i) \equiv z$. Consequently, we have an instance of *the double elevating property* specified in Table 7 below.

		k_1	k_1+1	k_1+2	k_1+3	
P''_i:	$\underbrace{\cdots\cdots}_{B(P''_i,a_1)=B(P'_i,a_1)}$	$\succ a_1$	$\succ x$	$\succ b_1$	$\succ \cdot$	$\succ \cdots$
P'_i:	$\overbrace{\underbrace{\cdots\cdots}_{B(P'_i,a_1)=B(P_i,b_1)}}$	$\succ a_1$	$\succ b_1$	$\succ x$	$\succ z$	$\succ \cdots$
P_i:	$\overbrace{\cdots\cdots}$	$\succ b_1$	$\succ a_1$	$\succ z$	$\succ x$	$\succ \cdots$

Table 7: An instance of *the double elevating property*

Hence, in each case, we induce a contradiction. Therefore, $b_1 \in \{r_{k_1}(P''_i), r_{k_1+1}(P''_i)\}$. In conclusion, we have either $r_{k_1}(P''_i) = a_1$ and $r_{k_1+1}(P''_i) = b_1$, or $r_{k_1}(P''_i) = b_1$ and $r_{k_1+1}(P''_i) = a_1$.

Last, we show $[r_{k_1}(P''_i) = a_1] \Rightarrow [B(P_i, b_1) = B(P'_i, a_1) = B(P''_i, a_1)]$, and $[r_{k_1}(P''_i) = b_1] \Rightarrow [B(P_i, b_1) = B(P'_i, a_1) = B(P''_i, b_1)]$. It is evident that $B(P_i, b_1) = B(P'_i, a_1)$.

We first assume $r_{k_1}(P''_i) = a_1$. Suppose $B(P'_i, a_1) \neq B(P''_i, a_1)$. Since $|B(P'_i, a_1)| = |B(P''_i, a_1)| = k_1 - 1$, $B(P'_i, a_1) \neq B(P''_i, a_1)$ implies that there exists $y \in B(P''_i, a_1)\backslash B(P'_i, a_1)$. Consequently, we have $y \; P''_i \; a_1$ and $a_1 \; P'_i \; y$ which by $P'_i \approx P''_i$ imply $a_1 \; P'_i! \; y$ and $y \; P''_i! \; a_1$. This contradicts the fact $r_{k_1}(P'_i) = r_{k_1}(P''_i) = a_1$. Therefore, $B(P_i, b_1) = B(P'_i, a_1) = B(P''_i, a_1)$.

We next assume $r_{k_1}(P''_i) = b_1$. Thus, $a_1 \; P'_i! \; b_1$ and $b_1 \; P''_i! \; a_1$. Suppose $B(P'_i, a_1) \neq B(P''_i, b_1)$. Since $|B(P'_i, a_1)| = |B(P''_i, b_1)| = k_1 - 1$, $B(P'_i, a_1) \neq B(P''_i, b_1)$ implies that there exists $y \in B(P''_i, b_1)\backslash B(P'_i, a_1)$. Consequently, we have $y \neq b_1$, $y \; P''_i \; b_1$ and $a_1 \; P'_i \; y$. Furthermore, since $b_1 \; P''_i \; a_1$, we have $y \; P''_i \; a_1$. Consequently, $P'_i \approx P''_i$ implies $a_1 \; P'_i! \; y$ and $y \; P''_i! \; a_1$. This contradicts the fact $a_1 \; P'_i! \; b_1$ and $b_1 \; P''_i! \; a_1$. Therefore, $B(P_i, b_1) = B(P'_i, a_1) = B(P''_i, b_1)$. This completes the verification of statement (i) for the pair (b_1, a_1).

Next, we provide an induction hypothesis: Given $1 < m \leqslant t$, statement (i) holds for all pairs $(b_1, a_1), \dots, (b_{m-1}, a_{m-1})$. We show that statement (i) holds for (b_m, a_m).

Since $a_m = r_{k_m}(P'_i)$ and $b_m = r_{k_m+1}(P'_i)$, $P'_i \approx P''_i$ implies $a_1 \in \{r_{k_m-1}(P''_i), r_{k_m}(P''_i), r_{k_m+1}(P''_i)\}$ and $b_1 \in \{r_{k_m}(P''_i), r_{k_m+1}(P''_i), r_{k_m+2}(P''_i)\}$. Suppose $a_m \notin \{r_{k_m}(P''_i), r_{k_m+1}(P''_i)\}$. Thus, $a_m = r_{k_m-1}(P''_i)$. Then, by $P'_i \approx P''_i$, we know (i) $r_{k_m}(P''_i) = r_{k_m-1}(P'_i) \equiv x$, (ii) $x \; P'_i! \; a_m$ and $a_m \; P''_i! \; x$, and (iii) $x \neq b_m$. Since $b_1 \in \{r_{k_m}(P''_i), r_{k_m+1}(P''_i), r_{k_m+2}(P''_i)\}$ and $x = r_{k_m}(P''_i)$, we know $b_1 \in \{r_{k_m+1}(P''_i), r_{k_m+2}(P''_i)\}$. We next assert $r_{k_m-1}(P_i) = x$. Suppose not, i.e., $r_{k_m-1}(P_i) \equiv y \neq x$. Since $r_{k_m}(P_i) = b_m \neq x$, $P_i \approx P'_i$ implies $r_{k_m-2}(P_i) = r_{k_m-1}(P'_i) = x$ and $r_{k_m-1}(P_i) = r_{k_m-2}(P'_i) = y$. Then, it must be the case $(b_{m-1}, a_{m-1}) = (x, y)$, and the induction hypothesis implies $x \in \{r_{k_m-2}(P''_i), r_{k_m-1}(P''_i)\}$ which contradicts $r_{k_m}(P''_i) = x$. Therefore, $r_{k_m-1}(P_i) = x$. Then, symmetric to the argument related to a_1 above, we have $B(P_i, x) = B(P'_i, x) = B(P''_i, a_m)$, and induce an instance of *the local elevating property* if $r_{k_m+1}(P''_i) = b_m$, and an instance of *the double elevating property* if $r_{k_m+2}(P''_i) = b_m$, which both contradict the hypothesis of Lemma 2. Hence, $a_m \in \{r_{k_m}(P''_i), r_{k_m+1}(P''_i)\}$.

Next, suppose $b_m \notin \{r_{k_m}(P''_i), r_{k_m+1}(P''_i)\}$. Thus, $b_m = r_{k_m+2}(P''_i)$. Then, by $P'_i \approx P''_i$, we know $r_{k_m+1}(P''_i) = r_{k_m+2}(P'_i) \equiv x$, and hence $x \neq a_m$. Since $a_m \in \{r_{k_m}(P''_i), r_{k_m+1}(P''_i)\}$ and $x = r_{k_m+1}(P''_i)$, we have $r_{k_m}(P''_i) = a_m$. Moreover, since $P_i \approx P'_i$ and $r_{k_m+1}(P_i) = a_m \neq x$, we know $x \in \{r_{k_m+2}(P_i), r_{k_m+3}(P_i)\}$. Then, symmetric to the argument related to b_1 above, we have $B(P_i, b_m) = B(P'_i, a_m) = B(P''_i, a_m)$, and induce an instance of *the local elevating property* if $r_{k_m+2}(P_i) = x$, and an instance of *the double elevating property* if $r_{k_m+3}(P_i) = x$, which both contradict the hypothesis of Lemma 2. Therefore, $b_m \in \{r_{k_m}(P''_i), r_{k_m+1}(P''_i)\}$. In conclusion, we have either either $r_{k_m}(P''_i) = a_m$ and $r_{k_m+1}(P''_i) = b_m$, or $r_{k_m}(P''_i) = b_m$ and $r_{k_m+1}(P''_i) = a_m$.

Last, symmetric to the argument related to (b_1, a_1) above, we assert $[r_{k_m}(P''_i) = a_m] \Rightarrow [B(P_i, b_m) = B(P'_i, a_m) = B(P''_i, a_m)]$, and $[r_{k_m}(P''_i) = b_m] \Rightarrow [B(P_i, b_m) = B(P'_i, a_m) = B(P''_i, b_m)]$. This completes the verification of the induction hypothesis. Hence, we prove statement (i) and Claim 15.

Last, since domain $\mathbb{D}$ is *weakly connected*, we know that every pair of distinct preferences is connected via a path. Then, Claim 15 immediately implies that all preferences of $\mathbb{D}$ are pairwise neighbors. Consequently, for an arbitrary pair of distinct preferences $P_i, P'_i \in \mathbb{D}$, if two objects are oppositely ranked across P_i and P_i', say $a\ P_i\ b$ and $b\ P_i'\ a$, they must be consecutively ranked in both P_i and P_i', i.e., $a = r_k(P_i) = r_{k+1}(P_i')$ and $b = r_{k+1}(P_i) = r_k(P_i')$ for some $1 \leqslant k < n$. Furthermore, by statements (i) and (ii) in the proof of Claim 15, we know that a and b also occupy the k-th and $(k+1)$-th ranking positions in every other preference of $\mathbb{D}$. Therefore, $\mathbb{D}$ must be a *restricted tier domain*. This completes the proof of Lemma 2.

D Details related to Remark 6

Let domain $\mathbb{D}$ satisfy *the local elevating property* (respectively, *the double elevating property*). We show that $\mathbb{D}$ admits no *sd-SP*, *sd-Eff* and *sd-EF* rule.

Assume w.l.o.g. that $\mathbb{D}$ includes preferences $\bar{P}_i, P_i$ and $\hat{P}_i$ of Table 1 (respectively, Table 2). Suppose that there exists an *sd-SP*, *sd-Eff* and *sd-EF* rule $\varphi : \mathbb{D}^n \to \mathcal{L}$. We construct two preference profiles which only consist of preferences $\bar{P}_i, P_i$ and $\hat{P}_i$ of Table 1 (respectively, Table 2):

- $P^1 \equiv (P_1, P_2, \ldots, P_{n-1}, \bar{P}_n)$: Agent n reports $\bar{P}_i$, while everyone else reports P_i.
- $P^2 \equiv (\hat{P}_1, P_2, \ldots, P_{n-1}, \bar{P}_n)$: Agent n reports $\bar{P}_i$, agent 1 reports $\hat{P}_i$, while everyone else reports P_i.

Step 1: For notational convenience, let $B \equiv B(\bar{P}_i, a) = B(P_i, a) = B(\hat{P}_i, b)$. Evidently, *sd-EF* and feasibility imply $\varphi_{iB}(P^1) = \frac{k-1}{n}$ and $\varphi_{iB}(P^2) = \frac{k-1}{n}$ for all $i \in I$. Then, by *sd-SP*, we have $\varphi_{1a}(P^1) + \varphi_{1b}(P^1) = \varphi_{1b}(P^2) + \varphi_{1a}(P^2)$.

Step 2: We specify the random assignments $\varphi(P^1)$ and $\varphi(P^2)$ over a and b by *sd-Eff* and *sd-EF* in the following two claims.

Claim 16 *The random assignment $\varphi(P^1)$ over a and b is specified below.*

	a	b
$1, \cdots, n-1:$	$\frac{1}{n}$	$\frac{1}{n-1}$
$n:$	$\frac{1}{n}$	0

First, *sd-EF* and feasibility imply $\varphi_{ia}(P^1) = \frac{1}{n}$ for all $i \in I$. Next, *sd-Eff* implies $\varphi_{nb}(P^1) = 0$. Then, by *sd-EF* and feasibility, we have $\varphi_{ib}(P^1) = \frac{1}{n-1}$ for all $i = 1, \ldots, n-1$. This completes the verification of the claim.

Claim 17 *The random assignment $\varphi(P^2)$ over a and b is specified below.*

	a	b
$1:$	0	$\frac{2n-3}{(n-1)^2}$
$2, \cdots, n-1:$	$\frac{1}{n-1}$	$\frac{n-2}{(n-1)^2}$
$n:$	$\frac{1}{n-1}$	0

First, *sd-Eff* implies $\varphi_{1a}(P^2) = 0$. Then, by *sd-EF* and feasibility, we have $\varphi_{ia}(P^2) = \frac{1}{n-1}$ for all $i = 2, \cdots, n$. Second, *sd-Eff* implies $\varphi_{nb}(P^2) = 0$. Then, feasibility implies $\varphi_{1b}(P^2) + \sum_{i=2}^{n-1} \varphi_{ib}(P^2) = 1$. Last, since *sd-EF* implies $\varphi_{1a}(P^2) + \varphi_{1b}(P^2) = \varphi_{ia}(P^2) + \varphi_{ib}(P^2)$ for all $i = 2, \cdots, n-1$, and $\varphi_{ib}(P^2) = \varphi_{jb}(P^2)$ for all $i, j \in \{2, \cdots, n-1\}$, we calculate $\varphi_{ib}(P^2) = \frac{n-2}{(n-1)^2}$ for all $i = 2, \cdots, n-1$, and $\varphi_{1b}(P^2) = \frac{2n-3}{(n-1)^2}$. This completes the verification of the claim.

Last, according to Claims 16 and 17, we have

$$\varphi_{1a}(P^1) + \varphi_{1b}(P^1) = \frac{1}{n} + \frac{1}{n-1} > \frac{2n-3}{(n-1)^2} = \varphi_{1b}(P^2) + \varphi_{1a}(P^2).$$

This contradicts Step 1. Therefore, $\mathbb{D}$ admits no *sd-SP*, *sd-Eff* and *sd-EF* rule.

E Proof of Lemma 3

Suppose that $\mathbb{D}(\Omega_K)$ satisfies *the local elevating property* (respectively, *the double elevating property*), including preferences $\bar{P}_i$, P_i and $\hat{P}_i$ of Table 1 (respectively, Table 2). Recall the algorithm. According to objects a, b and c in Table 1 (respectively, Table 2), there must exist a unique step $1 \leqslant k \leqslant K$ such that (i) a, b, c are included in a block $A_t \in \mathbf{A}_{k-1}$, and (ii) A_t breaks into A_t^1 and A_t^2 such that two objects of $\{a, b, c\}$ and the third one are separated in A_t^1 and A_t^2. We assume w.l.o.g. that two objects of $\{a, b, c\}$ are included in A_t^1, and the third one is included in A_t^2. Note that $\mathbb{D}(\Omega_K) \subseteq \mathbb{D}(\Omega_{K-1}) \subseteq \cdots \subseteq \mathbb{D}(\Omega_k)$, and in every preference of $\mathbb{D}(\Omega_k)$, either all objects of A_t^1 rank above all objects of A_t^2, or all objects of A_t^2 rank above all objects of A_t^1. Consequently, we have $[a, b \in A_t^1] \Rightarrow [\bar{P}_i \notin \mathbb{D}(\Omega_K)]$, $[a, c \in A_t^1] \Rightarrow [P_i \notin \mathbb{D}(\Omega_K)]$, and $[b, c \in A_t^1] \Rightarrow [\hat{P}_i \notin \mathbb{D}(\Omega_K)]$, which contradict the hypothesis of Lemma 3. Therefore, $\mathbb{D}(\Omega_K)$ avoids both *the local elevating* and *double elevating properties*.

F Proof of Proposition 1

We show that $\mathbb{D}(\Omega_K)$ is equivalent to a *sequentially dichotomous domain* of Liu (2019). Then, by Theorem 1 of Liu (2019), we know that the PS rule is *sd-SP* on $\mathbb{D}(\Omega_K)$ which hence completes the verification of Proposition 1.

We first introduce two new notions which are used in the definition of a *sequentially dichotomous domain*. First, a sequence of partitions $(\mathbf{A}_k)_{k=0}^T$ is called a *dichotomous path* if it satisfies the following two conditions:

1. $\mathbf{A}_0 = \{A\}$ and $\mathbf{A}_T = \big\{\{a\} : a \in A\big\}$.
2. For each $1 \leqslant k \leqslant T$, some $A_t \in \mathbf{A}_{k-1}$ breaks into nonempty A_t^1 and A_t^2, and $\mathbf{A}_k = \{A_t^1, A_t^2\} \cup \mathbf{A}_{k-1} \backslash \{A_t\}$.

Next, given a partition $\mathbf{A}_k = \{A_1, \ldots, A_t\}$, a preference P_i *respects* $\mathbf{A}_k$ if for distinct $A_p, A_q \in \mathbf{A}_k$, either all objects of A_p rank above all objects of A_q in P_i, or vice versa, i.e., either $a \; P_i \; b$ for all $a \in A_p$ and $b \in A_q$, or $b \; P_i \; a$ for all $a \in A_p$ and $b \in A_q$. Now, we introduce the definition of a *sequentially dichotomous domain*. A domain $\mathbb{D}$ is a **sequentially dichotomous domain** if there exists a dichotomous path $(\mathbf{A}_k)_{k=0}^T$ such that we have $\big[P_i \in \mathbb{D}\big] \Leftrightarrow [P_i$ respects all $\mathbf{A}_0, \ldots, \mathbf{A}_T]$.

Now, we start to prove Proposition 1. At each step of the algorithm, we generate a partition. Thus, we have $\mathbf{A}_0 = \{A\}$ and $\mathbf{A}_1, \ldots, \mathbf{A}_K$. At the termination step K, assume w.l.o.g. that $\mathbf{A}_K = \{A_1, \ldots, A_t, A_{t+1}, \ldots, A_{K+1}\}$ where $|A_k| = 2$ for all $k = 1, \ldots, t$, and $|A_s| = 1$ for all $s = t+1, \ldots, K+1$. Moreover, from $k = 1$ to $k = t$, we continue to consecutively break A_k into two singleton subsets, and dichotomously refine all restricted tier structures of Ω_K accordingly. Thus, for each $k = 1, \ldots, t$, we have partition $\mathbf{A}_{K+k}$ which replaces $A_k \in \mathbf{A}_{K+k-1}$ by the two corresponding singleton subsets, and the set of tier structures Ω_{K+k} which collects the *dichotomous refinements* of each tier structure in Ω_{K+k-1}. Thus, $\mathbf{A}_{K+t} = \big\{\{a\} : a \in A\big\}$, and we have a *dichotomous path* $\big(\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_K, \mathbf{A}_{K+1}, \ldots, \mathbf{A}_{K+t}\big)$. Now, to show that $\mathbb{D}(\Omega_K)$ is equivalent to the corresponding *sequentially dichotomous domain*, it suffices to show the following two conditions:

(i) Given $P_i \in \mathbb{D}(\Omega_K)$ and $0 \leqslant k \leqslant K + t$, P_i respects $\mathbf{A}_k$.

(ii) Given $P_i \notin \mathbb{D}(\Omega_K)$, there exists $0 \leqslant k \leqslant K + t$ such that P_i does not respect $\mathbf{A}_k$.

Given $P_i \in \mathbb{D}(\Omega_K)$, we know $P_i \in \mathbb{D}(\Omega_k)$ for all $k = 0, 1, \ldots, K$. Next, given $0 \leqslant k \leqslant K$, we know $P_i \in \mathbb{D}(\mathcal{P})$ for some $\mathcal{P} \in \Omega_k$. Then, Definition 2 implies that P_i respects $\mathbf{A}_k$. Therefore, P_i respects $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_K$. Furthermore, since blocks $A_1, \ldots, A_t$ of $\mathbf{A}_K$ consecutively break into two singleton subsets from $\mathbf{A}_K$ to $\mathbf{A}_{K+t}$, it is evident that P_i continues to respect $\mathbf{A}_{K+k}$ for all $k = 1, \ldots, t$. This completes the verification of condition (i).

Given $P_i \notin \mathbb{D}(\Omega_K)$, since $P_i \in \mathbb{P} \equiv \mathbb{D}(\Omega_0)$, there exists $0 < k \leqslant K$ such that $P_i \in \mathbb{D}(\Omega_{k-1})$ and $P_i \notin \mathbb{D}(\Omega_k)$. Accordingly, let $P_i \in \mathbb{D}(\mathcal{P})$ for some $\mathcal{P} \in \Omega_{k-1}$. Thus, P_i respects $\mathbf{A}_{k-1}$. Furthermore, assume that block $A_s \in \mathbf{A}_{k-1}$ breaks into A_s^1 and A_s^2, and let $\overline{\mathcal{P}}$ and $\underline{\mathcal{P}}$ be the corresponding *dichotomous refinements* of $\mathcal{P}$ in Step k of the algorithm. Since $0 < k \leqslant K$, it is true that $|A_s| \geqslant 3$, and either $|A_s^1| \geqslant 2$ or $|A_s^2| \geqslant 2$. Since $P_i \notin \mathbb{D}(\Omega_k)$, it is true that $P_i \notin \mathbb{D}(\overline{\mathcal{P}}) \cup \mathbb{D}(\underline{\mathcal{P}})$. Then, there exist $a, b \in A_s^1$ and $c \in A_s^2$, or $a, b \in A_s^2$ and $c \in A_s^1$ such that $a\ P_i\ c$ and $c\ P_i\ b$. Consequently, P_i does not respect $\mathbf{A}_k$. This completes the verification of condition (ii), and hence proves Proposition 1.

References

ABDULKADIROĞLU, A. AND T. SÖNMEZ (1998): "Random serial dictatorship and the core from random endowments in house allocation problems," *Econometrica*, 689–701.

——— (2003): "Ordinal efficiency and dominated sets of assignments," *Journal of Economic Theory*, 112: 157–172.

AKAHOSHI, T. (2014): "A necessary and sufficient condition for stable matching rules to be strategy-proof," *Social Choice and Welfare*, 43: 683–702.

ALTUNTAŞ, A. (2016): "Probabilistic assignment of objects when preferences are single-dipped," *mineo*.

AZIZ, H., F. BRANDL, AND F. BRANDT (2014): "On the incompatibility of efficiency and strategyproofness in randomized social choice." in *AAAI*, 545–551.

BARBERÀ, S., D. BERGA, AND B. MORENO (2012): "Domains, ranges and strategy-proofness: the case of single-dipped preferences," *Social Choice and Welfare*, 39: 335–352.

BARBERÀ, S., F. GUL, AND E. STACCHETTI (1993): "Generalized median voter schemes and committees," *Journal of Economic Theory*, 61: 262–289.

BIKHCHANDANI, S., S. CHATTERJI, R. LAVI, A. MU'ALEM, N. NISAN, AND A. SEN (2006): "Weak monotonicity characterizes deterministic dominant-strategy implementation," *Econometrica*, 74: 1109–1132.

BOGOMOLNAIA, A. (2015): "Random assignment: redefining the serial rule," *Journal of Economic Theory*, 158: 308–318.

BOGOMOLNAIA, A. AND H. MOULIN (2001): "A new solution to the random assignment problem," *Journal of Economic Theory*, 100: 295–328.

——— (2002): "A simple random assignment problem with a unique solution," *Economic Theory*, 19: 623–636.

CARROLL, G. (2012): "When are local incentive constraints sufficient?" *Econometrica*, 80: 661–686.

CHANG, H. I. AND Y. CHUN (2017): "Probabilistic assignment of indivisible objects when agents have the same preferences except the ordinal ranking of one object," *Mathematical Social Sciences*, 90: 80–92.

CHATTERJI, S. AND H. ZENG (2019): "Random mechanism design on multidimensional domains," *Journal of Economic Theory*, 182: 25–105.

CHO, W. J.(2016): "Incentive properties for ordinal mechanisms," *Games and Economic Behavior*, 95: 168–177.

——— (2016): "When is the probabilistic serial assignment uniquely efficient and envy-free?" *Journal of Mathematical Economics*, 66: 14–25.

——— (2018): "Probabilistic assignment: An extension approach," *Social Choice and Welfare*, 51(1): 137–162.

CRÈS, H. AND H. MOULIN (2001): "Scheduling with opting out: Improving upon random priority," *Operations Research*, 49: 565–577.

EHLERS, L. (2002): "Coalitional strategy-proof house allocation," *Journal of Economic Theory*, 105: 298–317.

GALE, D. AND L. S. SHAPLEY (1962): "College admissions and the stability of marriage," *American Mathematical Monthly*, 9–15.

GIBBARD, A. (1977): "Manipulation of schemes that mix voting with chance," *Econometrica*, 665–681.

HYLLAND, A. AND R. ZECKHAUSER (1979): "The efficient allocation of individuals to positions," *the Journal of Political Economy*, 293–314.

KANDORI, M., F. KOJIMA, AND Y. YASUDA (2010): "Tiers, preference similarity, and the limits on stable partners," *mimeo*.

KASAJIMA, Y. (2013): "Probabilistic assignment of indivisible goods with single-peaked preferences," *Social Choice and Welfare*, 41: 203–215.

KESTEN, O. (2009): "Why do popular mechanisms lack efficiency in random environments?" *Journal of Economic Theory*, 144: 2209–2226.

——— (2010): "School choice with consent," *the Quarterly Journal of Economics*, 125(3): 1297-1348.

KESTEN, O. AND M. KURINO (2013): "Do outside options matter in matching? A New Perspective on the Trade-offs in Student Assignment," *mimeo*.

KOJIMA, F. AND M. MANEA (2010): "Incentives in the probabilistic serial mechanism," *Journal of Economic Theory*, 145: 106–123.

LE BRETON, M. AND A. SEN (1999): "Separable preferences, strategyproofness, and decomposability," *Econometrica*, 67: 605–628.

LE BRETON, M. AND J. WEYMARK (1999): "Strategy-proof social choice with continuous separable preferences," *Journal of Mathematical Economics*, 32(1): 47–85.

LIU, P. (2019): "Strategy-proofness of the probabilistic serial rule on sequentially dichotomous domains," *Games and Economic Behavior*, accepted, http://ink.library.smu.edu.sg/soe_research/1915/.

MONJARDET, B. (2009): "Acyclic domains of linear orders: A survey," in *the Mathematics of Preference, Choice and Order*, Springer, 139–160.

MOULIN, H. (1980): "On strategy-proofness and single peakedness," *Public Choice*, 35: 437–455.

PÁPAI, S. (2000): "Strategyproof assignment by hierarchical exchange," *Econometrica*, 68: 1403–1433.

PUPPE, C. (2018): "The single-peaked domain revisited: A simple global characterization," *Journal of Economic Theory*, 176: 55–80.

PYCIA, M. AND M. U. ÜNVER (2017): "Incentive compatible allocation and exchange of discrete resources," *Theoretical Economics*, 12: 287–29.

SAPORITI, A. (2009): "Strategy-proofness and single-crossing," *Theoretical Economics*, 4: 127–163.

SATO, S. (2013): "A sufficient condition for the equivalence of strategy-proofness and nonmanipulability by preferences adjacent to the sincere one," *Journal of Economic Theory*, 148: 259–278.

SEN, A. K. (1966): "A possibility theorem on majority decisions," *Econometrica*, 491–499.

SHAPLEY, L. AND H. SCARF (1974): "On cores and indivisibility," *Journal of Mathematical Economics*, 1: 23–37.

SVENSSON, L.-G. (1999): "Strategy-proof allocation of indivisible goods," *Social Choice and Welfare*, 16: 557–567.

ZHOU, Y. AND S. SERIZAWA (2018): "Strategy-proofness and efficiency for non-quasi-linear and common-tiered-object preferences: Characterization of minimum price rule," *Games and Economic Behavior*, 109: 327-363.