Neighborhood Top Trading Cycles*

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Abstract

We study the classical reallocation problem (Shapley and Scarf, 1974) and introduce a large class of exchange rules, each of which is strategy-proof, efficient, and individually rational on the domain of single-peaked preferences. These rules are generalizations of Gale's top trading cycles: In each step, a subset of neighboring objects are available for exchange and the next subset of available objects may depend on the exchanges performed previously, where the neighborhood is defined by the order that justifies single-peakedness.

Keywords: Neighborhood; top trading cycles; strategy-proof; single-peaked;

JEL Classification: C78; D47

1 Introduction

We study the "reallocation problem" where some privately owned objects need to be reallocated to their owners and monetary transfer is not allowed for compensation. Each agent has a strict preference over objects and an exchange might be desirable. An exchange rule, or simply a rule, is hence a function which selects for each preference profile an allocation of objects to agents. For the reallocation problem, Shapley and Scarf (1974) prove that Gale's top trading cycles rule (Gale's TTC) always selects a core allocation, which implies efficiency and individual rationality. This rule is also strategy-proof (Roth, 1982). Moreover, if agents' preferences are strict but otherwise

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unrestricted, Gale's TTC is the unique rule satisfying strategy-proofness, efficiency, and individual rationality (Ma, 1994). [1]

However, if agents' preferences are "single-peaked", there are other rules satisfying the aforementioned properties. Consider for example that the objects to be reallocated are houses, each of which has been attached a price. Based on her financial status, an agent might have an optimal expense on housing and hence a house is better if and only if its price is closer to that optimal amount. For another example, houses are ordered by size. Each agent has an ideal size in mind and a house is better if and only if its size is closer to that ideal size. The single-peakedness of preferences has been wildly studied in various mechanism design problems. In particular, it was first introduced to the voting problem (Black, 1948; Moulin, 1980), and then to others such as the division problem (Sprumont, 1991) and the random assignment problem (Kasajima, 2013). More recently, for the reallocation problem with single-peaked preferences, Bade (2019) introduces a rule, called the "crawler", which is different from Gale's TTC but also satisfies the aforementioned properties.[2]

In this paper, we introduce a class of rules, called the "neighborhood TTCs", each of which satisfies strategy-proofness, efficiency, and individual rationality on the domain of single-peaked preferences. Logically, Gale's TTC and the crawler are special cases of our rules.

Given agents' preferences, Gale's TTC is implemented in the following way. Let every agent point to the owner of her favorite house. Since the number of agents is finite, there is at least one cycle, including self-cycles where an agent points to herself. We let every agent in a cycle get the house of the agent she points to and leave with her house. This procedure is repeated among the remaining agents, until no agent remains.

Our rules also attain an iteration structure. Each step is further divided into two sub-steps. The first sub-step is called a preparation sub-step, where we check for each remaining agent whether her current house is her favorite among the remaining ones. Let such agents leave with their current houses. We repeatedly do this until no agent's house is her favorite among the remaining ones. In the second sub-step, called an exchange sub-step, we conduct TTC exchanges. However, only the houses in a subset

[1] Since then, Gale's TTC is the leading rule for reallocation problems and it has been adapted to deal with other problems. Examples include the hierarchical exchange rules (Pápai, 2000) and the trading cycles rules (Pycia and Ünver, 2017) for the problem where objects are collectively owned and the you-request-my-house I-get-your-turn rules (Abdulkadiroğlu and Sönmez, 1999) for the problem where some objects are privately owned and the others are collectively owned. It has also been studied and compared to other rules for the two-sided problems where objects also have priorities on agents, for example, the school choice problem (Abdulkadiroglu and Sönmez, 2003) and the kidney exchange problem (Roth et al., 2004). Recently it has been adapted for the problem of trading fractional shares and probabilities (Kesten, 2009; Aziz, 2015; Altuntas and Phan, 2017).

[2] Recent studies, e.g., Tamura (2022) and Tamura and Hosseini (2022), present interesting results about the crawler.

of remaining ones are allowed to be exchanged among agents. We call this subset of houses an available set. Gale's TTC hence is a special case of our rules, where the available set in the first step contains all the houses remaining from the preparation sub-step. For details, please see Example 1.

By definition, each of our rules is parameterized by a tree structure that determines in each step the available set. We call such a parameter an availability tree and the correspondingly defined rule a generalized TTC. The available sets in a tree may be path-dependent, in the sense that the available set for a step is allowed to depend on the exchanges happened in previous steps.

Every generalized TTC satisfies efficiency and individual rationality on the unrestricted domain of preferences (Proposition 1). Ma (1994)'s characterization hence implies that such a rule is in general not strategy-proof if agents' preferences are unrestricted. Moreover, it turns out that, on the domain of single-peaked preferences, it is not true either that an arbitrary generalized TTC is strategy-proof (see Example 4). In particular, it is possible for an agent to misreport another preference so that, she takes an sacrifice in a specific iteration step, which eventually awards her with a better house. To eliminate such manipulations, we impose a restriction on the parameter of generalized TTCs, i.e., the tree structure that determines the sets of houses available for exchange. In particular, every such set of houses is required to be a neighborhood, meaning that these houses are adjacent by the order that justifies single-peakedness. We hence call such a generalized TTC a neighborhood TTC and prove that all such rules are strategy-proof on the domain of single-peaked preferences (Theorem 1).

One would probably agree that the set of trees satisfying the neighborhood restriction is quite large. Hence the class of neighborhood TTCs is large. We believe that the provision of such a large class of desirable rules could improve significantly the flexibility in practical applications. An example is provided, Example 3 in particular, where an agent requests an opportunity to exchange with a specific individual before the implementation of a full participation exchange.

The sequel of the paper is organized as follows. Section 2 introduces preliminary definitions and notations. Section 3 defines the neighborhood TTCs and presents their properties. Section 4 concludes.

2 Preliminaries

Let $I \equiv \{1, \cdots, n\}$ be the set of agents. Each agent $i \in I$ owns a house, denoted h_i. The set of houses is denoted $H \equiv \{h_1, \cdots, h_n\}$. A **sub-allocation** is a one-to-one mapping from a non-empty subset of agents $\hat{I}$ to a subset of houses $\hat{H}$ such that $|\hat{I}| = |\hat{H}|$. The set of all sub-allocations is denoted $\mathcal{M}$. Given an arbitrary sub-allocation $m \in \mathcal{M}$, $m(i)$ denotes the house allocated to i and $m^{-1}(h)$ the agent who gets h. The set of agents involved with sub-allocation $m \in \mathcal{M}$ is denoted I_m and set of houses involved H_m. For arbitrary group of agents $\hat{I} \subseteq I_m$, the set of houses allocated to them is denoted $m(\hat{I}) \equiv \{m(i) : i \in \hat{I}\}$. Similarly, for an arbitrary $\hat{H} \subseteq H_m$, we write $m^{-1}(\hat{H}) \equiv \{m^{-1}(h) : h \in \hat{H}\}$. An **allocation** is a sub-allocation $m \in \mathcal{M}$ with full participation, i.e., it's a mapping from I to H. We denote the set of allocations by M. Hence $M \subset \mathcal{M}$. The **endowment** is evidently an allocation $e \in M$ such that $e(i) = h_i$ for all $i \in I$.

Each agent $i \in I$ is equipped with a strict preference P_i on houses, i.e., an antisymmetric, transitive, and complete binary relation on H.[3] For an arbitrary preference P_i and an arbitrary nonempty subset of houses $\hat{H} \in 2^H \backslash \{\emptyset\}$, let $\tau(P_i, \hat{H})$ denote i's favorite house in $\hat{H}$, i.e., $\tau(P_i, \hat{H}) = h$ such that $h\ P_i\ h'$ for all $h' \in \hat{H} \backslash \{h\}$. In particular, let $\tau(P_i) \equiv \tau(P_i, H)$. Let $\mathcal{P}$ be the set of all strict preferences. We call $\mathcal{P}$ the universal domain. For a specific allocation problem, the set of admissible preferences might not be the universal domain but a subset $\mathcal{D} \subseteq \mathcal{P}$. We call such a subset the **domain** of admissible preferences.

An **economy** is a tuple (I, H, P, e), where $P = (P_i)_{i \in I} \in \mathcal{D}^n$ is a profile of admissible preferences, one for each agent. Throughout the paper we fix I, H, and e. Hence we denote an economy simply by $P \in \mathcal{D}^n$. An exchange rule, or simply a **rule**, is a mapping $\varphi : \mathcal{D}^n \to M$ which selects for each economy an allocation.

We focus on the domain of single-peaked preferences. To define these preferences, a linear order on H is fixed. Such a linear order is denoted $<$. Without loss of generality, we fix it as $h_1 < h_2 < \cdots < h_n$ throughout the paper. We interpret this order as the ranking of houses according to size and say h is smaller than h' if $h < h'$. In addition, we write $h \leqslant h'$ if either h is smaller than h' or they refer to the same house. A preference $P_i \in \mathcal{P}$ is single-peaked with respect to $<$ if h is preferred to h' whenever h is closer to the favorite house according to $<$. Formally,

Definition 1. *A preference $P_i \in \mathcal{P}$ is **single-peaked** with respect to $<$ if, $\forall h, h' \in H$,*

$$h' < h \leqslant \tau(P_i) \Rightarrow h\ P_i\ h', \text{ and } \tau(P_i) \leqslant h < h' \Rightarrow h\ P_i\ h'.$$

The **single-peaked domain** is hence defined as the set of *all* preferences which are single-peaked with respect to $<$. We denote it $\mathcal{D}_<$.

A rule is desirable if it satisfies the axioms below.

The first axiom deals with incentive compatibility. It requires that reporting true preference in the direct revelation game is always a weakly dominant strategy.

Definition 2. *A rule $\varphi : \mathcal{D}^n \to M$ is **strategy-proof** if and only if, for all $P \in \mathcal{D}^n$ and all $P_i' \in \mathcal{D}$, $\varphi_i(P) \neq \varphi_i(P_i', P_{-i})$ implies $\varphi_i(P)\ P_i\ \varphi_i(P_i', P_{-i})$.*

The second axiom requires that the chosen allocation can not be improved in the way that some agent gets a better house without any other agent getting a worse house.

[3] A binary relation P_i is antisymmetric if $h\ P_i\ h'$ and $h'\ P_i\ h$ imply $h = h'$, transitive if $h\ P_i\ h'$ and $h'\ P_i\ h''$ imply $h\ P_i\ h''$, and complete if either $h\ P_i\ h'$ or $h'\ P_i\ h$ holds for arbitrary h and h'.

Definition 3. *For an arbitrary economy, $P \in \mathcal{D}^n$, an allocation $m \in M$ is **efficient** if and only if there exists no $m' \in M$ such that $m' \neq m$ and, for all $i \in I$, $m'(i) \neq m(i) \Rightarrow m'(i)\ P_i\ m(i)$. A rule is efficient if it selects for each economy an efficient allocation.*

The third axiom requires that no agent gets a house worse than her endowment.

Definition 4. *For an arbitrary economy, $P \in \mathcal{D}^n$, an allocation $m \in M$ is **individually rational** if and only if, for all $i \in I$, $m(i) \neq h_i \Rightarrow m(i)\ P_i\ h_i$. A rule is individually rational if it selects for each economy an individually rational allocation.*

3 Neighborhood Top Trading Cycles

We define in this section our rules and present their properties. To begin with, we define some preliminary notions.

First, we generalize Gale's TTC by restricting the set of houses available for exchange. To do so, let $P \in \mathcal{D}^n$ be an arbitrary economy and $m \in \mathcal{M}$ an arbitrary sub-allocation, which is treated as the status quo. Moreover, let $\hat{H} \subseteq H_m$ be an arbitrary nonempty subset of houses. These houses are the ones available for exchange. Then $TTC(P, m, \hat{H}) \equiv \hat{m}$ is the allocation resulted by applying TTC algorithm to the agents in $m^{-1}(\hat{H})$. In particular, $\hat{m} : I_m \to H_m$ is specified below.

- $\forall i \in I_m \backslash m^{-1}(\hat{H})$, $\hat{m}(i) = m(i)$;
- $\forall i \in m^{-1}(\hat{H})$, $\hat{m}(i)$ is specified by the following iteration.
  - Round 1: Every agent in $m^{-1}(\hat{H})$ points to the agent whose house is her favorite in $\hat{H}$, i.e., agent $i \in m^{-1}(\hat{H})$ points to j such that $\tau(P_i, \hat{H}) = m(j)$. Since $m^{-1}(\hat{H})$ is finite, there must be a cycle. Then each agent in a cycle gets the house of the agent she points to. Let these agents leave with their houses.
  - Round $2, 3, \cdots$: If there are agents remaining from previous rounds, let each of them point to the agent whose house is her favorite among the remaining ones. There must be at least one cycle. Then each agent in a cycle gets the house of the agent she points to. Let these agents leave with their houses.
  - Iteration terminates when there is no agent remaining.

Second, we say a sub-allocation $m \in \mathcal{M}$ is nested in another sub-allocation $m' \in \mathcal{M}$ if $H_m \subseteq H_{m'}$ and $I_m \subseteq I_{m'}$. A sequence of sub-allocations $m_1 m_2 \cdots$ is called a **history** if it satisfies the following two conditions.

1. $m_1(i) = h_i, \forall i \in I_{m_1}$.

2. m_{k+1} is nested in m_k for all $k = 1, 2, \cdots$.

The first condition is equivalent to saying that m_1 is the endowment e restricted to a subset of agents (or equivalently a subset of houses). The set of all histories is denoted Γ. The length of a history $\gamma \in \Gamma$ is the number of sub-allocations in it and denoted $|\gamma|$. Let $\bar{\Gamma} \equiv \{\gamma \in \Gamma : |\gamma| = +\infty\}$ be the set of all infinite histories. We call them the terminal histories. Moreover, we say γ is a sub-history of γ' and denote it $\gamma \subseteq \gamma'$ if γ' is created by appending some sub-allocations to γ. For example, $m_1 m_2 m_3$ is a sub-history of itself and $m_1 m_2 m_3 m_4$.

Third, given the fixed order of houses $<$ and an arbitrary sub-allocation, a **neighborhood** refers to a subset of adjacent houses. Formally, a nonempty subset of houses $\hat{H}$ is a neighborhood with respect to $<$ and $m \in \mathcal{M}$, if for all $h, h', h'' \in H_m$ such that $h' < h < h''$, $[h', h'' \in \hat{H}] \Rightarrow [h \in \hat{H}]$. Given an arbitrary pair $h' \leqslant h''$, we denote the neighborhood $\{h \in H : h' \leqslant h \leqslant h''\}$ simply $[h', h'']$. By convention, (h', h''), $(h', h'']$, $[h', h'')$ are analogously defined. It should be noted that, a subset of houses that is a neighborhood with respect to a sub-allocation may not be a neighborhood with respect to another sub-allocation. For example, $\{h_1, h_3, h_4\}$ is a neighborhood with respect to a sub-allocation $m : \{3, 4, 5, 7\} \rightarrow \{h_1, h_3, h_4, h_6\}$. However, it's not a neighborhood with respect to the endowment e. Moreover, for any sub-allocation m, H_m is a neighborhood.

Fourth, an **availability tree**, or simply a tree, is a function $T : \Gamma \backslash \bar{\Gamma} \rightarrow 2^H \backslash \{\emptyset\}$ satisfying the following two conditions.

1. $\forall \gamma \in \Gamma \backslash \bar{\Gamma}$, $T(\gamma) \subseteq H_m$ where m is the last sub-allocation in γ.
2. $\forall \bar{\gamma} \in \bar{\Gamma}$, there is a non-terminal sub-history $\gamma \subseteq \bar{\gamma}$ such that $T(\gamma) = H_m$ where m is the last sub-allocation in γ.

The availability tree functions as the preset mechanism that determines at every history the subset of houses available for exchange. Hence the first condition above is a nature requirement. The second condition implies that in finitely many steps, all remaining houses are available for exchange. This condition ensures that our rules are well-defined, i.e., for any preference profile, an allocation is determined in finitely many steps. For our purpose, we are interested in a particular type of trees, the ones where all available sets are neighborhoods. Formally, $T : \Gamma \backslash \bar{\Gamma} \rightarrow 2^H \backslash \{\emptyset\}$ is called a **neighborhood tree** if, for all $\gamma \in \Gamma \backslash \bar{\Gamma}$, $T(\gamma)$ is a neighborhood with respect to $<$ and m, where m is the last sub-allocation in γ.

Last, given an arbitrary economy $P \in \mathcal{D}^n$ and an arbitrary sub-allocation $m \in \mathcal{M}$, $\overline{m}_P$ denotes the maximal sub-allocation where no agent's house is her favorite among $H_{\overline{m}_P}$. Hence $\overline{m}_P$ is generated by excluding repeatedly the agents whose houses are their favorite. Put otherwise, $\overline{m}_P$ identities the agents who have motivation to participate in exchange and also the houses available for exchange in effect. When P is clearly announced, we simply denote $\overline{m}_P$ by $\overline{m}$.

We are now ready to define our rules.

Definition 5. *Fix an availability tree $T : \Gamma\backslash\bar{\Gamma} \to 2^H\backslash\{\emptyset\}$. A* ***generalized TTC*** *is an exchange rule $\varphi^T : \mathcal{D}^n \to M$, where for every $P \in \mathcal{D}^n$, $\varphi^T(P)$ is resulted from the following iteration.*

- ***Step*** 1*:*
  - *Preparation sub-step: Check for each agent whether her current house is her favorite among the remaining ones. If so, let this agent leave with her current house. We repeatedly do so until no such agent exists. If no agent remains, iteration terminates. Otherwise, denote the resulted sub-allocation by $\overline{e}$.*
  - *Exchange sub-step: Run TTC and let $m_1 \equiv TTC(P, \overline{e}, T(\overline{e}))$.*
- ***Step*** $k = 2, 3, \cdots$*:*
  - *Preparation sub-step: Check for each agent whether her current house is her favorite among the remaining ones. If so, let this agent leave with her current house. We repeatedly do so until no such agent exists. If no agent remains, iteration terminates. Otherwise, denote the resulted sub-allocation by $\overline{m}_{k-1}$.*
  - *Exchange sub-step: Run TTC and let $m_k \equiv TTC(P, \overline{m}_{k-1}, T(\overline{e} \cdots \overline{m}_{k-1}))$.*

When T is a neighborhood tree, we call φ^T a ***neighborhood TTC****.*

By the definition of availability tree, for any preference profile, all houses are available for exchange in finitely many steps. Moreover, in the preparation sub-step next to such a step, all remaining agents leave the procedure. Hence, our rule selects an allocation in finitely many steps. Before discussions on other properties of our rules, we present below two examples.

Example 1. *Let T be such that $T(m) = H_m$ for every $m \in \Gamma\backslash\bar{\Gamma}$. Put otherwise, all houses are available for exchange in the first step. Then the neighborhood TTC $\varphi^T : \mathcal{D}^n \to M$ is equivalent to Gale's TTC.* ■

Example 2. *Bade (2019) proposes an exchange rule, which is strategy-proof on the single-peaked domain. This rule is called the "crawler" and implemented step by step. In each step, we identify the first agent, from the one in the smallest house to the one in the largest, whose favorite house is no larger than her own. If her own house is her favorite, let her leave with her own house. Otherwise, let the agent be i and let her favorite house be h_j. Then $h_j < h_i$. Let agent i get h_j and leave. For each agent whose house is in $[h_j, h_i)$, let her "crawl" to a larger house next to her own. Exactly* *one agent leaves with a house in each step and hence the procedure terminates at the n-th step.*

The crawler appears different from TTC algorithm. However, it is equivalent to the neighborhood TTC defined by a particular tree. To show this tree, for an arbitrary subset of houses $\hat{H}$, let $r_t(\hat{H})$ be the t-th smallest house. For example, $r_1(\hat{H})$ is the smallest house and $r_{|\hat{H}|}(\hat{H})$ is the largest.

Let $T : \Gamma\backslash\bar{\Gamma} \to 2^H\backslash\{\emptyset\}$ be such that (i) $T(m) = \{r_{|H_m|}(H_m), r_{|H_m|-1}(H_m)\}$ contains the two largest houses for every $m \in \Gamma\backslash\bar{\Gamma}$, and (ii) for $\gamma = m_1 m_2, \cdots, m_K$, $K \geqslant 2$

$$T(\gamma) = \begin{cases} \left\{r_{|H_{m_K}|}(H_{m_K}), r_{|H_{m_K}|-1}(H_{m_K})\right\}, & H_{m_K} \neq H_{m_{K-1}} \\ \{r_{t-1}(H_{m_K}), r_{t-2}(H_{m_K})\}, & \begin{array}{l} H_{m_K} = H_{m_{K-1}} \text{ where } t \text{ is s.t.} \\ T(m_1 \cdots m_{K-1}) = \{r_t(H_{m_{K-1}}), r_{t-1}(H_{m_{K-1}})\} \end{array} \end{cases}$$

Put otherwise, after the preparation round in the first step, if there are agents remaining, let the largest two houses be available for exchange. The available set in the next step depends on whether the remaining houses are different from those in the previous step. If so, the largest two houses are available for exchange. Otherwise, the available houses are the two largest ones, each smaller than one in the previous step. For example, if $\{h_1, h_2, h_4, h_7\}$ are the houses at hand and the available set in the previous step is $\{h_4, h_7\}$, then $\{h_2, h_4\}$ is the available set in the current step.

We illustrate this neighborhood TTC by applying it to the Example 1 in Bade (2019), where $I = \{1, \cdots, 7\}$ and $H = \{h_1, \cdots, h_7\}$. Agents' preferences are single-peaked with respect to $h_1 < h_2 < \cdots < h_7$ such that $\tau(P_2) = h_2$, $\tau(P_4) = h_6$, $\tau(P_6) = h_3$, $\tau(P_7) = h_5$, and $\tau(P_i) = h_7$ for all others.

- *Step 1:*
  - *Preparation sub-step: Agent 2 leaves with h_2 and the resulted sub-allocation is below.*
$$\bar{e} = \begin{pmatrix} 1 & 3 & 4 & 5 & 6 & 7 \\ h_1 & h_3 & h_4 & h_5 & h_6 & h_7 \end{pmatrix}$$
  - *Exchange sub-step: Run TTC*
$$m_1 = TTC(P, \bar{e}, T(\bar{e})) = TTC(P, \bar{e}, \{h_6, h_7\}) = \bar{e}.$$
- *Step 2:*
  - *Preparation sub-step: No one leaves and the resulted sub-allocation is $\overline{m}_1 = \bar{e}$.*
  - *Exchange sub-step: Run TTC*
$$\begin{aligned} m_2 &= TTC(P, \overline{m}_1, T(\overline{em}_1)) = TTC(P, \overline{m}_1, \{h_5, h_6\}) \\ &= \begin{pmatrix} 1 & 3 & 4 & 6 & 5 & 7 \\ h_1 & h_3 & h_4 & h_5 & h_6 & h_7 \end{pmatrix}. \end{aligned}$$

- *Step 3:*
  - *Preparation sub-step: No one leaves and the resulted sub-allocation is* $\overline{m}_2 = m_2$.
  - *Exchange sub-step: Run TTC*

$$\begin{aligned} m_3 &= TTC(P, \overline{m}_2, T(\overline{em}_1 \overline{m}_2)) = TTC(P, \overline{m}_2, \{h_4, h_5\}) \\ &= \begin{pmatrix} 1 & 3 & 6 & 4 & 5 & 7 \\ h_1 & h_3 & h_4 & h_5 & h_6 & h_7 \end{pmatrix}. \end{aligned}$$

- *Step 4:*
  - *Preparation sub-step: No one leaves and the resulted sub-allocation is* $\overline{m}_3 = m_3$.
  - *Exchange sub-step: Run TTC*

$$\begin{aligned} m_4 &= TTC(P, \overline{m}_3, T(\overline{em}_1 \overline{m}_2 \overline{m}_3)) = TTC(P, \overline{m}_3, \{h_3, h_4\}) \\ &= \begin{pmatrix} 1 & 6 & 3 & 4 & 5 & 7 \\ h_1 & h_3 & h_4 & h_5 & h_6 & h_7 \end{pmatrix}. \end{aligned}$$

- *Step 5:*
  - *Preparation sub-step: Agent* 6 *leaves with* h_3 *and the resulted sub-allocation is below.*

$$\overline{m}_4 = \begin{pmatrix} 1 & 3 & 4 & 5 & 7 \\ h_1 & h_4 & h_5 & h_6 & h_7 \end{pmatrix}$$

  - *Exchange sub-step: Run TTC*

$$\begin{aligned} m_5 &= TTC(P, \overline{m}_4, T(\overline{em}_1 \overline{m}_2 \overline{m}_3 \overline{m}_4)) = TTC(P, \overline{m}_4, \{h_6, h_7\}) \\ &= \begin{pmatrix} 1 & 3 & 4 & 7 & 5 \\ h_1 & h_4 & h_5 & h_6 & h_7 \end{pmatrix}. \end{aligned}$$

The remaining steps are left for the reader, who should already be able to see the equivalence. In particular, a "crawl" in Bade (2019)'s rule is implemented in the above by three 2-agent swaps, i.e., the exchanges in steps 2 to 4.[4] ■

Example 3. *Imagine that a reallocation of offices will be conducted among faculty members and Gale's TTC is supposed to be employed. Before the implementation, agent 2 requests to seek the opportunity for an exchange with agent 3 beforehand. This is probably because agent* 2 *wants to get* 3*'s office and she realized the possibility*

[4] Schummer and Serizawa (2019) propose an exchange rule, called the "iterative swap algorithm", which can also be seen as a special case of neighborhood TTCs. Recently, Huang (2022) studies a class of exchange rules called the r-neighborhood mechanisms. The relationship between these rules and the neighborhood TTCs defined in the current paper is discussed in section 4.2 in Huang (2022).

that Gale's TTC may not let her do so. Then, we may consider the neighborhood TTC defined by the availability tree below.

$$T(m) = [h_2, h_3]; \forall m \in \Gamma \backslash \bar{\Gamma} \text{ s.t. } [h_2, h_3] \subset H_m$$
$$T(\gamma) = H_m, \forall |\gamma| > 2 \text{ where } m \text{ is the last sub-allocation in } \gamma$$

In particular, if agents 2 *and* 3 *remain from the first preparation sub-step, they are allowed to swap there offices. After that, Gale's TTC will be implemented.*

■

For any generalized TTC, whenever an agent exchanges her house with others, she is strictly better-off. Hence, individual rationality is evident. Moreover, efficiency can be seen by the argument that proves the efficiency of Gale's TTC: for each agent, the house she gets is the best among the ones not assigned to others yet. Hence, whenever we improve an agent's welfare by giving her a better house, there must be some other agent who is worse off. We hence have the following.

Proposition 1. *Every generalized TTC is efficient and individually rational on the universal domain.*

The next question we are interested in is whether every generalized TTC is strategy-proof on the single-peaked domain. The example below gives an answer in the negative.

Example 4. *Let* $I = \{1, 2, 3\}$ *and* $H = \{h_1, h_2, h_3\}$. *Fix the order* $<$ *such that* $h_1 < h_2 < h_3$. *Then all the preferences below are single-peaked with respect to* $<$.

$$P_1: \; h_2 \succ h_3 \succ h_1 \qquad P_3: \; h_1 \succ h_2 \succ h_3$$
$$P_2: \; h_1 \succ h_2 \succ h_3 \qquad P_1': \; h_2 \succ h_1 \succ h_3$$

Consider the economies $P = (P_1, P_2, P_3)$ *and* $P' = (P_1', P_2, P_3)$ *where agent* 1 *unilaterally deviates. Let* T *be an availability tree such that* $T(e) = \{h_1, h_3\}$ *and, for all other histories* γ, $T(\gamma) = H_m$ *where* m *is the last sub-allocation in* γ.

- *Step 1:*
  - *Preparation sub-step: No one leaves and the resulted sub-allocation is* $\bar{e} = e$.
  - *Exchange sub-step: Run TTC*

$$m_1 = TTC(P, \bar{e}, T(\bar{e})) = TTC(P, \bar{e}, \{h_1, h_3\}) = \begin{pmatrix} 3 & 2 & 1 \\ h_1 & h_2 & h_3 \end{pmatrix}.$$

- *Step 2:*

- *Preparation sub-step: Agent 3 leaves with h_1 which is her favorite house. Then agent 2 leaves with h_2 since it is her favorite among the houses excluding h_1. Thereafter, agent 1 leaves with h_3 and the iteration terminates.*

If agent 1 reports P_1'. The iteration steps are as follows.

- *Step 1:*
  - *Preparation sub-step: No one leaves and the resulted sub-allocation is $\overline{e} = e$.*
  - *Exchange sub-step: Run TTC*

$$m_1 = TTC(P', \overline{e}, T(\overline{e})) = TTC(P', \overline{e}, \{h_1, h_3\}) = \overline{e}.$$

- *Step 2:*
  - *Preparation sub-step: No one leaves and the resulted sub-allocation is $\overline{m}_1 = e$.*
  - *Exchange sub-step: Run TTC*

$$m_2 = TTC(P', \overline{m}_1, T(\overline{em}_1)) = TTC(P', \overline{em}_1, H) = \begin{pmatrix} 2 & 1 & 3 \\ h_1 & h_2 & h_3 \end{pmatrix}.$$

- *Step 3:*
  - *Preparation sub-step: Agents 1 and 2 leave with respectively h_2 and h_1. Then agent 3 also leaves. Hence iteration terminates.*

Since $\varphi_1^T(P') = h_2$ is better than $\varphi_1^T(P) = h_3$ according to agent 1' true preference, she has an incentive to manipulate the procedure. Specifically, through such a manipulation, agent 1 holds her own house h_1, rather than exchanging it with agent 3 for h_3 in the first step. Only by doing so, can she exchange with agent 2 for h_2 (agent 1's favorite) in the next step, since agent 2 wants only h_1. Put otherwise, agent 1 makes a sacrifice temporarily, which leads to a better outcome eventually.

■

One may note that $\{h_1, h_3\}$ in Example 4 is not a neighborhood. The theorem below states that if the availability tree is a neighborhood tree, there is no such manipulation anymore.

Theorem 1. *Every neighborhood TTC on the single-peaked domain is strategy-proof, efficient, and individually rational.*

By Proposition 1, we need only to prove strategy-proofness, which is in Appendix A. In particular, let an arbitrary agent unilaterally change her preference. We identify the first step when this agent's house is different. Let it be the change from getting h (under truth-telling) to getting h' (with unilateral deviation) in that step. By the definition of step-wise TTC, it is evident that h is better than h' according to the agent's true preference. We prove that, no matter what are the subsequent available sets, the house this agent will get eventually, with unilateral deviation, is no better than h'. Note that, under truth-telling, the agent's final allocation is no worse than h. Hence, strategy-proofness is verified. It's interesting to note (see the proof) that, it is the neighborhood restriction and the single-peakedness which rule out the manipulations as the one in Example 4. Put otherwise, no agent can get a better outcome by making a temporary sacrifice.

Given Theorem 1, two natural questions are as follows. First, is it true that every strategy-proof, efficient, and individually rational rule on the single-peaked domain is a neighborhood TTC? Second, is single-peakedness necessary for the neighborhood TTCs to satisfy these axioms? These two questions are interesting but difficult to answer, especially because of the fact that the set of neighborhood TCCs is large. We hence reserve them for further studies. However, the proposition below studies the second question from anther angle: the single-peaked domain is maximal for the class of neighborhood TTCs to be strategy-proof. Specifically, whenever a non-single-peaked preference is made admissible in addition to the single-peaked domain, there exists a manipulable neighborhood TTC.

Proposition 2. *For any non-single-peaked preference $P_0 \in \mathcal{P} \backslash \mathcal{D}_<$, there is a neighborhood tree T such that $\varphi^T : (\mathcal{D}_< \cup \{P_0\})^n \to M$ is not strategy-proof.*

The proof is in Appendix B. We close this section by the remarks below.

Remark 1. *Given Proposition 1, one may ask, given a preference profile, whether every efficient and individually rational allocation is the result of some generalized TTC? The answer is in the negative, as shown by the example below. Consider $I = \{1, 2, 3\}$ and $H = \{h_1, h_2, h_3\}$. Below are agents' preferences and a specific allocation m.*

$$\begin{array}{ll} P_1 : & h_2 \succ h_3 \succ h_1 \\ P_2 : & h_1 \succ h_2 \succ h_3 \\ P_3 : & h_2 \succ h_3 \succ h_1 \end{array} \qquad m = \begin{pmatrix} 2 & 3 & 1 \\ h_1 & h_2 & h_3 \end{pmatrix}$$

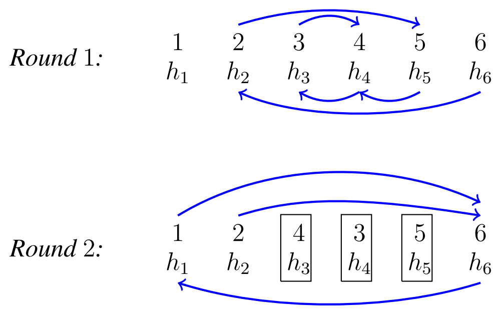
It's evident that m is individually rational and efficient at (P_1, P_2, P_3). However, there is no tree T such that $\varphi^T(P) = m$. To see this, note that there are in this example four effective available sets: (i) $\{h_1, h_2, h_3\}$, (ii) $\{h_1, h_2\}$, (iii) $\{h_2, h_3\}$, and (iv) $\{h_1, h_3\}$. In the first step, if the available set is (i), agent 1 and 2 will exchange with each other. Since they get their favorite houses, they will not exchange anymore, no matter what is the remaining part of the tree. Hence the final allocation can not be m. If the first available set is (ii), agent 1 and 2 will exchange with each other and, similar with case (i), the final allocation can not be m. For the other two cases, there will be no exchange in the first step. Then, we move to the second step. If the available set is either one of (i) and (ii), the final allocation is not m. Otherwise, we move to the *third step. Finally, by the definition of tree, in a finite step, all houses are available and hence the final allocation is not m.* ■

Remark 2. *Due to the fact that step-wise TTC is itself an iteration, one may suspect that, strategy-proofness is preserved if we further allow the available sets to be different for different rounds, within a step-wise TTC. By doing so, one can enlarge the class of strategy-proof rules. Unfortunately, it's wrong, as shown by the example below.*

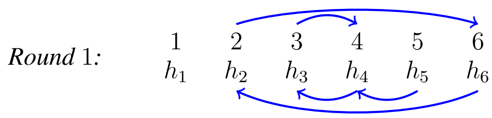
Consider $I = \{1, 2, 3, 4, 5, 6\}$ and $H = \{h_1, h_2, h_3, h_4, h_5, h_6\}$. Agents' preferences are single-peaked with respect to $h_1 < h_2 < h_3 < h_4 < h_5 < h_6$, as below.

$$\begin{array}{llll} P_1: & h_6 \succ \cdots & P_4: & h_3 \succ h_4 \succ \cdots \\ P_2: & h_5 \succ h_6 \succ \cdots & P_5: & h_4 \succ h_5 \succ \cdots \\ P_3: & h_4 \succ h_3 \succ \cdots & P_6: & h_1 \succ \cdots \end{array}$$

Suppose the available set for the first round is $[h_2, h_6]$, after which Gale's TTC is implemented. Then, one can verify that agent 2's final allocation is h_2. The procedure of exchanges is illustrated below. In particular, in the first round, houses in $[h_2, h_6]$ are available for exchange. Then agent 3 and 4 form a cycle and swap their houses. Thereafter, agents 3 and 4 leave with their new houses which are their favorite ones. Observing this, agent 5 also leaves with h_5 since she wants only h_4 which has been taken by 3. In the second round when all houses are available for exchange, agent 1 and 6 form a cycle. Hence agent 2 stays in her own house in the final allocation.



Exchanges under (P_2, P_{-2})



Exchanges under (P'_2, P_{-2})

However, if agent 2 reports a false preference $P'_2 = P_1$. In particular, she says that her peak is h_6 rather than h_5. Then in the first round when houses in $[h_2, h_6]$ are available for exchange, she gets h_6, which is better than h_2 under truth-telling.

In the exmaple above, since agent 1 is not allowed to exchange her house in the beginning, agent 2 has a chance to get h_6. Realizing that agent 5 treats h_2 unacceptable, agent 2 can make a compromise that she points to agent 6 rather than 5. The reason that agent 2 needs to make such a decision is because, from round 1 to round 2, the available set changes. Hence she has only one chance to point to h_6. However, if the available set stays unchanged, as defined by our neighborhood TTCs, agent 2 is allowed to try pointing to the owner of every house, from the best to the worst. Hence such a manipulation is impossible for our rules. ■

4 Conclusion

This paper presents a large class of exchange rules, which are strategy-proof, efficient, and individually rational on the single-peaked domain. Gale's TTC (Shapley and Scarf, 1974) and the crawler (Bade, 2019) are special cases of our rules. We believe that, equipped with the flexibility provided by such a large class of desirable rules, the mechanism designer is now able to deal with more context-specific requirements. Two relevant questions are as follows. First, is it true that the class of neighborhood TTCs are characterized by the axioms aforementioned? Second, is single-peakedness necessary for these rules to satisfy the aforementioned axioms?

Appendix

A Proof of Theorem 1

Fix an arbitrary neighborhood TTC $\varphi^T : \mathcal{D}^n_< \to M$. For the purpose of proving strategy-proofness, we present below two observations.

Observation 1. *Fix an arbitrary preference profile $P \in \mathcal{D}^n_<$. Let*

$$TTC_i(P, m_{k-1}, T(m_1 \cdots m_{k-1})) = h.$$

In the procedure of TTC, if at the round when agent i gets h, two other houses h' and h'' haven't yet been taken by any agent such that $h' < h < h''$, then $\varphi^T_i(P) = h$.

Proof. By definition, the available set in step k is a neighborhood containing h' and h''. We abuse notation a bit by using $[h', h'']$ to denote the houses in $H_{m_{k-1}} \cap T(m_1 \cdots m_{k-1})$ that are ranked in between h' and h'', including two boundaries. Analogously, we define (h', h''). We partition the houses in (h', h'') by timing of being taken by agents. In particular, let $H^1 \subseteq (h', h'')$ be the set of houses taken by agents in the earliest round. Similarly, let $H^2 \subseteq (h', h'') \backslash H^1$ be the set of houses, excluding H^1, taken by agents in the earliest round. In this way, we partition (h', h'') into finitely many subsets H^1, H^2, $\cdots$, H^S. Moreover, we denote the sets of agents who get these houses as I^1, I^2, $\cdots$, I^S. In the round when an agent $j \in I^1$ gets a house in H^1, all the houses in $[h', h'']$ are available and she points to a house other than the extreme ones, i.e., h' and h''. Single-peakedness then implies that the house she gets in this round is exactly her favorite in $[h', h'']$. Hence all agents in I^1 will leave with their houses in the preparation round of next step. Similarly, in the round when an agent $j \in I^2$ gets a house in H^2, all the houses in $[h', h''] \backslash H^1$ are available and she points to a house other than the extreme ones. Then single-peakedness implies that the house she gets in this round is her favorite in $[h', h''] \backslash H^1$. Hence agents in I^2 will leave with their houses in the preparation round of next step. Since agent i is contained in a particular set $I^s \in \{I^1, \cdots, I^S\}$, the domino-like argument above implies that agent i will leave with h in the preparation round of the next step, i.e., $\varphi_i^T(P) = h$. ■

Observation 2. *Fix an arbitrary preference profile $P \in \mathcal{D}_<^n$. For any agent i and step k such that $m_{k-1}(i) \in T(m_1 \cdots m_{k-1})$, let $h = TTC_i(P, m_{k-1}, T(m_1 \cdots m_{k-1}))$. If $m_{k-1}(i) < \tau(P_i) \leqslant h$, then then $\varphi_i^T(P) = h$.*

Proof. The observation is evident if $\tau(P_i) = h$. We hence verify the case where $\tau(P_i) < h$. First, $m_{k-1}(i) < \tau(P_i) < h$ and single-peakedness imply that, for every house h_j such that $h_j P_i h$, (i) $m_{k-1}(i) < h_j < h$ and (ii) h_j has been taken by some agent j in an earlier round. One may note that, such an agent j gets h_j in a round when $m_{k-1}(i)$ and h are available such that $m_{k-1}(i) < h_j < h$. Then Observation 1 implies that such an agent j's final allocation is h_j. Consequently, we have the conclusion. ■

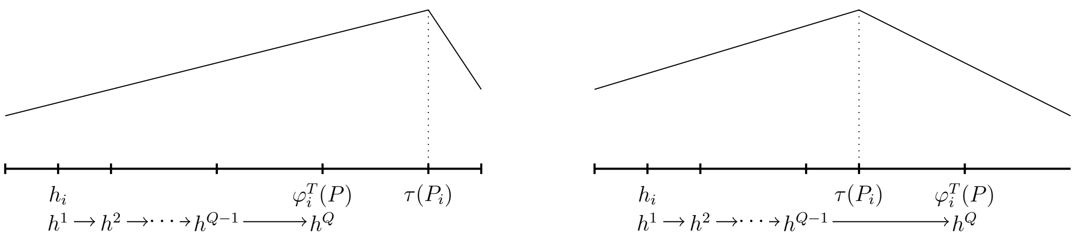
To prove the strategy-proofness of φ^T, we fix an arbitrary preference profile and an arbitrary agent i who is the unilateral deviator. Without loss of generality, we assume that agent i's favorite house is larger than her endowment, i.e., $h_i < \tau(P_i)$. We track agent i's houses in the procedure, as follows.

$$h_i \equiv h^1 \to h^2 \to \cdots \to h^q \to h^{q+1} \to \cdots \to h^{Q-1} \to h^Q \equiv \varphi_i^T(P).$$

In particular, agent i's endowment is h_i, which is denoted h^1. In the first step where h^1 is available for exchange, agent i's house changes to be h^2. We keep track of her houses in all steps where her house is available for exchange, until her final allocation $\varphi_i^T(P)$, which is denoted h^Q. Note that, h^q could be the same as h^{q+1} in the sequence. Given observations above, we are able to depict this sequence, as in Lemma 1 below.

Lemma 1.

- *If $\varphi_i^T(P) \leqslant \tau(P_i)$, then $h^1 \leqslant h^2 \leqslant \cdots \leqslant h^Q = \varphi_i^T(P)$.*
- *If $\varphi_i^T(P) > \tau(P_i)$, then $h^1 \leqslant \cdots \leqslant h^{Q-1} < \tau(P_i) < h^Q = \varphi_i^T(P)$.*



(a) $\varphi_i^T(P) \leqslant \tau(P_i)$

(b) $\varphi_i^T(P) > \tau(P_i)$

Figure 1: The sequence of agent i's houses.

Proof. The case where $\varphi_i^T(P) \leqslant \tau(P_i)$ is evident by single-peakedness and the fact that whenever agent i exchanges with other agents, she gets a better house. As for the other case where agent i's house changes in some step from a house smaller than her peak to one larger than the peak, Observation 2 implies that this house is exactly her final allocation.

■

Given agent i's sequence of houses, we investigate the implications of her unilateral deviation from P_i to an arbitrary preference $P_i' \in \mathcal{D}_<$. We identify the first divergence between the sequence of agent i's houses at (P_i, P_{-i}) and the sequence at (P_i', P_{-i}). In particular, let it be a particular step k where agent i changes her house from h^q to h^{q+1} at P and to $h' \neq h^{q+1}$ at (P_i', P_{-i}). Note that $\varphi_i^T(P)$ is no worse than h^{q+1}. Hence, to prove $\varphi_i^T(P)\ P_i\ \varphi_i^T(P_i', P_{-i})$, it suffices to show $h^{q+1} P_i \varphi_i^T(P_i', P_{-i})$. To do so, note first that, since all other agents' preferences hold constant and agent i's sequence of houses before the k-th step remain the same, the sub-histories at (P_i, P_{-i}) and (P_i', P_{-i}) before the k-th step are the same. Hence the available set in the k-th step remains unchanged.

By Lemma 1, in particular the facts that $h^{q+1} \geqslant h^q$ and that if $h^q \geqslant \tau(P_i)$ then $h^q = h^{q+1}$, either one of the following four cases is true.

Case 1: $h^q < h^{q+1} \leqslant \tau(P_i)$;

Case 2: $h^q = h^{q+1} \leqslant \tau(P_i)$;

Case 3: $h^q < \tau(P_i) < h^{q+1}$;

Case 4: $h^q = h^{q+1} > \tau(P_i)$.

Lemmas 2 to 5 prove $h^{q+1} P_i \varphi_i^T(P_i', P_{-i})$ for each of these cases.

Lemma 2. *If $h^q < h^{q+1} \leqslant \tau(P_i)$, then $h^{q+1}\ P_i\ \varphi_i^T(P_i', P_{-i})$.*

Proof. Given $h' \neq h^{q+1}$, the strategy-proofness and individual rationality of TTC imply either $h' \in [h^q, h^{q+1})$ or $h' \in [\tilde{h}', \tilde{h}'']$ where $\tilde{h}'$ and $\tilde{h}''$ are on the other side of the peak such that $\tilde{h}'$ is the smallest house that is no better than h^{q+1} and $\tilde{h}''$ is the largest house that is no worse than h^q, as illustrated in Figure 2.

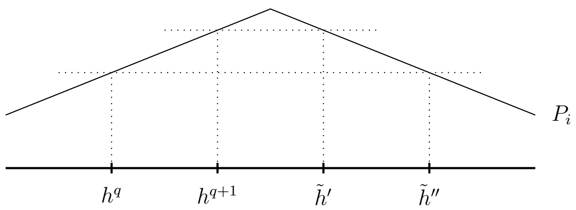


Figure 2: Illustration of Lemma 2.

If $h' \in [\tilde{h}', \tilde{h}'']$, since $h' > h^q$, Lemma 1 implies that $\varphi_i^T(P_i', P_{-i}) \geqslant h'$ and hence $h^{q+1}\ P_i\ \varphi_i^T(P_i', P_{-i})$.

If $h' \in [h^q, h^{q+1})$, we show first $\tau(P_i') \in [h^q, h^{q+1})$. Suppose otherwise $\tau(P_i') \geqslant h^{q+1}$. Note that P_i' restricted to $[h^q, h^{q+1}]$ is the same as P_i, which implies that agent i will never point to a house smaller than h^{q+1} whenever h^{q+1} is available and hence $h' \geqslant h^{q+1}$: contradiction. In the case where $\tau(P_i') \leqslant h^q$. Lemma 1 implies $\varphi_i(P_i', P_{-i}) \leqslant h^q$ and hence $h^{q+1}\ P_i\ \varphi_i^T(P_i', P_{-i})$. We hence assume $\tau(P_i') \in (h^q, h^{q+1})$. In this case, if $h' \geqslant \tau(P_i')$, Lemma 1 implies $\varphi_i^T(P_i', P_{-i}) = h'$ and hence $h^{q+1}\ P_i\ \varphi_i^T(P_i', P_{-i})$. If instead $h' < \tau(P_i')$. Suppose $h^{q+1}\ P_i\ \varphi_i^T(P_i', P_{-i})$ is not true. Then single-peakedness implies $\varphi_i^T(P_i', P_{-i}) \geqslant h^{q+1}$. By definition of φ^T, $\varphi_i^T(P_i', P_{-i})\ P_i'\ h'$ and hence $h^{q+1}\ P_i'\ h'$. However if $h^{q+1}\ P_i'\ h'$, agent i should not get h' because she can always get h^{q+1} by pointing to its owner and form a cycle with others: contradiction. ■

Lemma 3. *If $h^q = h^{q+1} \leqslant \tau(P_i)$, then $h^{q+1}\ P_i\ \varphi_i^T(P_i', P_{-i})$.*

Proof. If $\tau(P_i') \leqslant h^q$, the conclusion is evident. We hence assume $\tau(P_i') > h^q$. The strategy-proofness of TTC then implies that $h' \geqslant \tilde{h}'$ where $\tilde{h}'$ is on the other side of the peak $\tau(P_i)$ and is the smallest house that is no better than h^{q+1}. Figure 3 below illustrates the situation. Lemma 1 then implies $\varphi_i^T(P_i', P_{-i}) \geqslant \tilde{h}'$ and hence $h^{q+1}\ P_i\ \varphi_i^T(P_i', P_{-i})$. ■

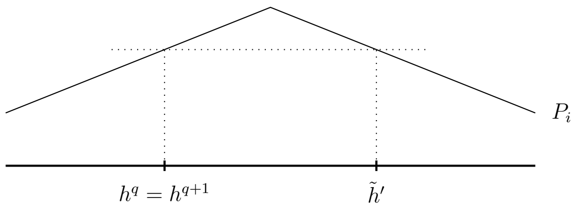


Figure 3: Illustration of Lemma 3.

Lemma 4. *If $h^q < \tau(P_i) < h^{q+1}$, then $h^{q+1}\ P_i\ \varphi_i^T(P_i', P_{-i})$.*

Proof. Given $h' \neq h^{q+1}$, the strategy-proofness and individual rationality of TTC imply either $h' \in [h^q, \tilde{h}'']$ or $h' \in (h^{q+1}, \tilde{h}']$ where $\tilde{h}''$ is the largest house such that $\tilde{h}'' < h^{q+1}$ and $h^{q+1}\ P_i\ \tilde{h}''$ and $\tilde{h}'$ is the smallest house such that $\tilde{h}' > h^q$ and $\tilde{h}'\ P_i\ h^q$. Figure 4 below illustrates the situation.

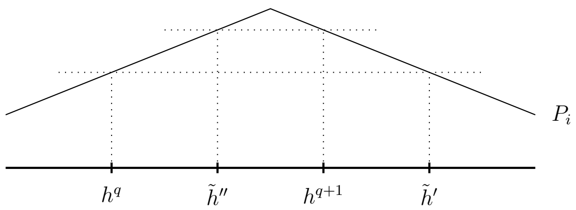


Figure 4: Illustration of Lemma 4.

If $h' \in (h^{q+1}, \tilde{h}']$, Lemma 1 implies $\varphi_i^T(P_i', P_{-i}) \geqslant h^{q+1}$ and hence $h^{q+1}\ P_i\ \varphi_i^T(P_i', P_{-i})$.

Suppose $h' \in [h^q, \tilde{h}'']$. If $\tau(P_i') \leqslant h^q$, Lemma 1 implies $\varphi_i^T(P_i', P_{-i}) \leqslant h^q$ and hence $h^{q+1}\ P_i\ \varphi_i^T(P_i', P_{-i})$. If $\tau(P_i') \geqslant h^{q+1}$, P_i and P_i' restricted to $[h^q, h^{q+1}]$ are the same. Then agent i should not get h' because she can always get h^{q+1} by pointing to its owner and form a cycle with others: contradiction. We hence assume $h^q < \tau(P_i') < h^{q+1}$. If $h^{q+1}\ P_i'\ h'$, agent i should not get h' for the same reason aforementioned. Hence $h'\ P_i'\ h^{q+1}$. Suppose, agent i gets h^{q+1} in the t-th round under (P_i, P_{-i}) and she gets h' in the t'-th round under (P_i', P_{-i}). Note that the preferences of agents other than i remain unchanged. If $t' \geqslant t$, the cycles formed before the t-th round are the same under (P_i, P_{-i}) and (P_i', P_{-i}). Hence, under (P_i', P_{-i}), every house in $U(P_i, h^{q+1})$ is taken by an agent when h^q and h^{q+1} are available. Observation 1 then implies that these agents will not exchange in future steps and hence $h^{q+1}\ P_i\ \varphi_i^T(P_i', P_{-i})$. If otherwise, $t' < t$. In the t'-th round under (P_i', P_{-i}), all houses in $U(P_i', h')$ have been taken by others in the previous rounds. Note that, every such agent gets a house in $U(P_i', h')$ when h^q and h^{q+1} are available. Then Observation 1 implies that these agent will not exchange in future steps and hence $\varphi_i^T(P_i', P_{-i}) = h'$. Then we have $h^{q+1}\ P_i\ \varphi_i^T(P_i', P_{-i})$. ■

Lemma 5. *If $h^q = h^{q+1} > \tau(P_i)$, then $h^{q+1}\ P_i\ \varphi_i^T(P_i', P_{-i})$.*

Proof. Figure 5 below illustrates the situation. By our assumption $h_i < \tau(P_i)$ and

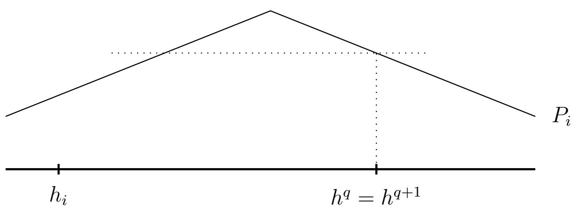


Figure 5: Illustration of Lemma 5.

Lemma 1, $\varphi_i^T(P_i, P_{-i}) = h^q$. If $\tau(P_i') \leqslant h^q$, Lemma 1 implies $h' = h^{q+1}$: contradiction. If otherwise $\tau(P_i') > h^q$, Lemma 1 implies $\varphi_i^T(P_i', P_{-i}) > h^{q+1}$ and hence $h^{q+1}\ P_i\ \varphi_i^T(P_i', P_{-i})$. ■

B Proof of Proposition 2

Let $P_0 \in \mathcal{P} \backslash \mathcal{D}_<$ be an arbitrary preference that is not single-peaked with respect to $<$. Let $h_k = \tau(P_0)$, then there are h_i, h_j such that $h_i < h_j < h_k$ and $h_i\ P_0\ h_j$. (The symmetric case where $h_k < h_j < h_i$ can be handled similarly.) We define as follows a neighborhood tree. To do so, consider the following two sub-allocations.

$$m = \begin{pmatrix} i & j & k \\ h_i & h_j & h_k \end{pmatrix} \quad m' = \begin{pmatrix} i & j & k \\ h_j & h_i & h_k \end{pmatrix}$$

Let T be such that $T(m) = \{h_i, h_j\}$, $T(mm) = T(mm') = \{h_j, h_k\}$, and for any other history γ, $T(\gamma) = \mu$ where μ is the last sub-allocation in γ. Let $\varphi^T : (\mathcal{D}_< \cup \{P_0\})^n \to M$ be the accordingly defined neighborhood TTC. Let in addition $P \in (\mathcal{D}_< \cup \{P_0\})^n$ be a profile of preferences such that $P_i = P_j = P_0$, $P_k \in \mathcal{D}_<$ such that $h_j \; P_k \; h_k \; P_k \; h$ for all $h \neq h_j, h_k$. For the agents $l \neq i, j, k$, let $P_l \in \mathcal{D}_<$ such that $e(l) = \tau(P_l)$. Figure 6 below illustrates the preferences.

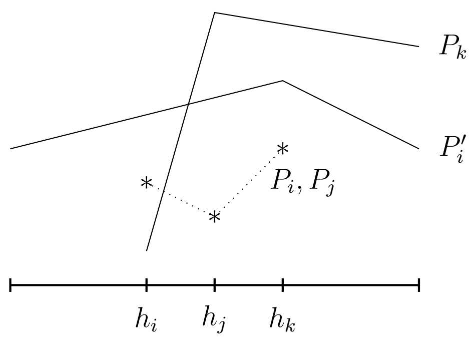


Figure 6: Illustration of the preferences in Proof of Proposition 2.

At economy P, the iteration steps are as follows.

- Step 1:
  - Preparation sub-step: All agents except for i, j, k leave with their own houses. The resulted sub-allocation is m.
  - Exchange sub-step: Run TTC $m_1 = TTC(P, m, T(m)) = TTC(P, m, \{h_i, h_j\}) = m$.
- Step 2:
  - Preparation sub-step: No agent leaves and the resulted sub-allocation is still m.
  - Exchange sub-step: Run TTC
    $$m_2 = TTC(P, m, T(mm)) = TTC(P, m, \{h_j, h_k\}) = \begin{pmatrix} i & k & j \\ h_i & h_j & h_k \end{pmatrix}.$$
- Step 3:
  - Preparation sub-step: Agents j and k leave with their houses and then i leaves h_i. Hence iteration terminates.

At economy $P' = (P_i', P_{-i})$, the iteration steps are as follows.

- Step 1:

- Preparation sub-step: All agents except for i, j, k leave with their own houses. The resulted sub-allocation is m.
- Exchange sub-step: Run TTC $m_1 = TTC(P', m, T(m)) == TTC(P', m, \{h_i, h_j\}) = m'$.

- Step 2:
  - Preparation sub-step: No agent leaves and the resulted sub-allocation is m'.
  - Exchange sub-step: Run TTC
    $m_2 = TTC(P', m', T(mm')) = TTC(P', m, \{h_j, h_k\}) = \begin{pmatrix} j & k & i \\ h_i & h_j & h_k \end{pmatrix}$.

- Step 3:
  - Preparation sub-step: Agents i and k leave with their houses and then j also leaves. Hence iteration terminates.

By the above, $\varphi_i^T(P) = h_i$ and $\varphi_i^T(P_i', P_{-i}) = h_k$. By agent i's true preference $h_k P_i h_i$. Hence φ^T is not strategy-proof.

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