Presenting Objects for Random Allocation *

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Abstract

For random allocation, whether a desirable rule exists hinges on the domain of agents' preferences, whose formation is affected by how objects are presented. We hence propose a model studying how to present objects so that the induced preference domain allows for designing a good rule. Motivated by practices in reality, we model the objects as combinations of several attribute values and a presentation of objects concerns a choice of presenting attributes and a ranking of them. Agents are assumed to formulate their preferences in a lexicographic manner according to the given presentation. We show that, the domain of preferences induced by a presentation allows for a strategy-proof, efficient, and envy-free rule if and only if the presented attributes are conditionally binary. Under two technical conditions on the number of objects, this result still holds when envy-freeness is weakened to equal treatment of equals.

Keywords: Random allocation; strategy-proof; efficiency; envy-free; equal treatment of equals; presenting;

JEL Classification: D02, C78.

1 Introduction

A general lesson we have learned from studies on mechanism design is that the existence of desirable mechanisms hinges on the domain of agents' preferences. Take for instance the voting problem, where the celebrated Gibbard-Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975) states that, if agents' preferences are unrestricted, a unanimous and strategy-proof voting rule must be dictatorial. However, if their preferences are 'single-peaked', all median voting rules satisfy the axioms aforementioned (Moulin, 1980). Recently, a similar observation has been recorded for the random allocation problem. In particular, if agents' preferences are unrestricted, there does not exist a random allocation rule satisfying jointly strategy-proofness,

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efficiency, and fairness (Bogomolnaia and Moulin, 2001). While, if agents' preferences are 'sequentially dichotomous', the so-called probabilistic serial rule satisfies all three axioms aforementioned (Liu, 2020).

As for the formation of agents' preferences, evidences from psychology and marketing (the so-called framing effect for example) indicate that how objects are presented affects agents' preferences.[1] A commonly seen presentation in reality is one where each object is presented as a sequence of attribute values. Take for example the HDB program in Singapore, which is a large scale public housing project accommodating around 80 percent of Singapore residents. Each flat type is presented as a list of the number of bedrooms, the number of bathrooms, the approximate floor area, etc. Moreover, it is suggested that, to choose a type of flats for application, an applicant should first decide on the number of bedrooms needed, followed by the number of bathrooms needed, and so on.

The current paper hence studies the question of how to present the objects for random allocation so that the induced preferences of agents allow for designing a good allocation rule. Motivated by observation from reality, we model the presentation as a choice of a subset of attributes and a ranking of them. Moreover, we assume that agents formulate their preferences in a lexicographic manner, according to the presented ranking of attributes. For example, when flats are presented as the bedroom number and then the bathroom number, an applicant figures out her preference on flats in the following way. First, she formulates a preference of feasible bedroom numbers and treat a flat better if and only if it contains a more preferred number of bedrooms. In this way, she has essentially a coarse preference over flats: they are partitioned into groups and a preference over these groups is determined. Next, within every such group, she further formulates a preference on the bathroom numbers that are conditionally feasible. Then the preference over flats in this group can be determined accordingly. Example 1 illustrates different presentations and the corresponding domains of lexicographic preferences.

Our results show that the unique class of presentations fulfilling our purpose are conditionally binary, meaning that, in every lexicographic step, the group of objects at hand is further partitioned into two smaller groups. In particular, Theorem 1 states the uniqueness for allocation rules satisfying strategy-proofness, efficiency, and envy-freeness. As a fairness axiom, envy-freeness requires that every agent treats her own lottery over objects weakly better than all others'. It is well-recorded in the literature of random allocation that envy-freeness is much stronger than another fairness axiom, called equal treatment of equals, requiring that whenever two agents have the same preference they receive the same lottery over objects. We are hence interested in the possibility of identifying more presentations by weakening envy-freeness to equal treatment of equals. However, Theorem 2 gives an answer in the negative, provided that the problem size, i.e., the number of objects, satisfies two technical conditions.[2]

For practice in reality, our results can serve as a support for presenting objects as a list of attributes that are conditionally binary. In this sense, our results are in line with quite a

[1]One may refer to Salant and Rubinstein (2008) for a theoretic analysis with the framing effect. As for estimating how do consumers' preferences depend on products' attributes, one can easily find many relevant studies in the marketing literature, e.g., Colonna et al. (2011).

[2]The conditions are two inequalities concerning the problem size. We have verified the conditions numerically for all problem sizes no larger than 3000. More detailed discussion can be found in Remark 2.

few studies in decision theory. For instance, Mandler et al. (2012) show that choosing by sequentially checking a list of binary attributes provides a rapid procedural basis for possibly complicated utility maximization problems.

The remainder of the paper is organized as follows. Section 2 introduces the formal model of random allocation and defines presentations and the induced lexicographic preferences. Section 3 presents results and Section 4 concludes. The appendix gathers proofs omitted.

2 Model and Preliminary Definitions

Subsection 2.1 introduces the random allocation problem. Subsection 2.2 defines the presentation of objects and the induced preference domain.

2.1 Random Allocation Problem

Let $I \equiv \{1, \cdots, n\}$ be the set of agents and O the set of objects. We assume $|I| = |O| = n \geqslant 4$ and each agent is supposed to get one object.[3] We call n the size of problem. Each agent i is endowed with a strict preference P_i on O, i.e., a complete, transitive and antisymmetric binary relation on O. Let $\mathcal{P}$ denote the set of *all* such preferences. For a specific allocation problem, some preferences may not be admissible. We hence denote the set of admissible preferences by $\mathcal{D}$, which is a subset of $\mathcal{P}$ and referred to as the preference domain. Given $P_i \in \mathcal{D}$, let $r_k(P_i)$ denote the k-th ranked object according to P_i. A preference profile $P \equiv (P_1, \ldots, P_n) \in \mathcal{D}^n$ is an n-tuple of admissible preferences, one for each agent. By convention, P_{-i} denotes the collection of agents' preferences excluding a particular agent i. When illustrating a particular preference P_0, we write $o \succ o'$ to express that o is strictly preferred to o'. Analogously, for two nonempty subsets of objects O_1 and O_2, we write $O_1 \succ O_2$ to express the fact that every object in O_1 is strictly preferred to every object in O_2. Moreover, we say the objects in a subset $O' \subset O$ are adjacent in a preference P_0 if, $\forall o, o', o''$ such that $o \succ o' \succ o''$, $o, o'' \in O'$ implies $o' \in O'$. Finally, we call a subset of objects a block if they are adjacent in a preference.

In pursuit of (ex ante) fairness, we introduce randomization. In particular, let $\Delta(O)$ denote the set of lotteries over objects, i.e., the probability distributions on the set O. Given $\lambda \in \Delta(O)$, λ_o denotes the probability of getting a particular object o. A random **allocation**, or simply an allocation, is a bi-stochastic matrix $L \equiv [L_{io}]_{i\in I, o\in O}$, namely a non-negative square matrix whose elements in each row add up to one and whose elements in each column also add up to one. Formally, (i) $L_{io} \geqslant 0$ for all $i \in I$ and $o \in O$, (ii)$\sum_{o\in O} L_{io} = 1$ for all $i \in I$, and (iii) $\sum_{i\in I} L_{io} = 1$ for all $o \in O$. Evidently, in an allocation L, the i-th row is the lottery agent i receives. By the celebrated Birkhoff-von Neumann theorem, every allocation can be implemented as a lottery over deterministic allocations that are ex post feasible, where each agent gets exactly one object. Let $\mathcal{L}$ denote the set of all allocations.

[3]Situations where the number of agents differs from that of objects can be easily modified into our model setting, by introducing null agents or null objects. An important class of allocation problems where our model does not apply are the ones where an agent might get more than one object. Those problems have been shown critically different from the problems we study. For specific discussions, one may refer to Budish (2011), Budish and Cantillon (2012), and Chatterji and Liu (2020), among others.

Agents assess lotteries according to first-order stochastic dominance. In particular, given a preference $P_i \in \mathcal{D}$ and two lotteries $\lambda, \lambda' \in \Delta(O)$, λ stochastically dominates λ' according to P_i, denoted $\lambda\ P_i^{sd}\ \lambda'$, if λ assigns to every upper contour set of objects a probability no lower than that assigned by λ'. Formally, $\sum_{l=1}^{k} \lambda_{r_l(P_i)} \geqslant \sum_{l=1}^{k} \lambda'_{r_l(P_i)}$ for all $1 \leqslant k \leqslant n$. Analogously, given $P \in \mathcal{D}^n$, we say an allocation L stochastically dominates L' according to P, denoted $L\ P^{sd}\ L'$, if $L_i\ P_i^{sd}\ L'_i$ for all $i \in I$. Note that, $\lambda\ P_i^{sd}\ \lambda'$ is equivalent to the requirement that, for *any* Bernoulli utility representing P_i, λ generates an expected utility that is no lower than that generated by λ'.

An allocation rule, or simply a rule, is a mapping $\varphi : \mathcal{D}^n \to \mathcal{L}$ which selects an allocation for every profile of admissible preferences. Given $P \in \mathcal{D}^n$, $\varphi_i(P)$ denotes the lottery agent i receives and $\varphi_{io}(P)$ denotes her probability to get a particular object $o \in O$.

We impose three axioms on a desirable rule. First, a rule $\varphi : \mathcal{D}^n \to \mathcal{L}$ is **strategy-proof** if, for every agent, it's a weakly dominant strategy to report her true preference in the corresponding direct revelation game. Formally, $\forall\ i \in I$, $P_i, P'_i \in \mathcal{D}$, and $P_{-i} \in \mathcal{D}^{n-1}$, $\varphi_i(P_i, P_{-i})\ \ P_i^{sd}\ \ \varphi_i(P'_i, P_{-i})$. Second, a rule $\varphi : \mathcal{D}^n \to \mathcal{L}$ is **efficient** if, $\forall\ P \in \mathcal{D}^n$, $\varphi(P)$ is Pareto optimal. Formally, $\forall P \in \mathcal{D}^n$ and $L \in \mathcal{L}$, $L\ P^{sd}\ \varphi(P)$ implies $L = \varphi(P)$. For fairness, we study two notions: equal treatment of equals and envy-freeness, where the latter is stronger than the former. In particular, a rule $\varphi : \mathcal{D}^n \to \mathcal{L}$ satisfies **equal treatment of equals** if whenever two agents have the same preference, they receive the same lottery. Formally, $\forall$ $P \in \mathcal{D}^n$ and $i, j \in I$, $P_i = P_j$ implies $\varphi_i(P) = \varphi_j(P)$. **Envy-freeness** requires that every agent finds her own lottery weakly better than all others'. Formally, $\forall P \in \mathcal{D}^n$ and $i, j \in I$, $\varphi_i(P) P_i^{sd} \varphi_j(P)$.

The probabilistic serial rule (PS for short) is introduced by Bogomolnaia and Moulin (2001) and has been proven efficient and envy-free. For a formal definition of this rule, please refer to the original paper. We describe here how it operates at an arbitrary preference profile. In particular, it treats objects as if they are infinitely divisible and operates as follows. All agents 'eat' their respectively favorite object at the uniform speed, until some object is exhausted. Thereafter, agents turn to 'eat' their respectively favorite objects among the remaining ones until some other object is exhausted. This procedure repeats until all objects are exhausted. Finally, the share of an object eaten by an agent is interpreted as the probability that this agent gets that particular object. By Bogomolnaia and Moulin (2001), we have the following lemma.

Lemma 1. *The probabilistic serial rule is efficient and envy-free on any preference domain.*

Unfortunately, the PS rule is not strategy-proof on the unrestricted preference domain $\mathcal{P}$. Put otherwise, it is possible for an agent to gain from reporting a false preference. Intuitively, the possibility of profitable manipulation is provided by the fact that a unilateral change in preference may change dramatically the eating agenda. However, it turns out that such manipulations are impossible on some restricted domains, among which are the 'sequentially dichotomous domains' (SDDs for short) by Liu (2020). We hence have the lemma below.[4]

Lemma 2. *The probabilistic serial rule is strategy-proof on the sequentially dichotomous domains.*

[4]The definition of sequentially dichotomous domains can be found in Appendix A.

2.2 Presentation of Objects and the Induced Preference Domain

Let $\mathcal{A}$ be a set of attributes and, for each attribute $A \in \mathcal{A}$, V_A denotes the set of feasible values. Every object to be allocated can be evaluated according to each attribute. Formally, $\forall o \in O$ and $A \in \mathcal{A}$, o_A denotes the corresponding attribute value. Consequently, the set of objects O is now a subset of Cartesian product $\prod_{A\in\mathcal{A}} V_A$. Without loss of generality, we assume no redundant attribute value. Put otherwise, for every attribute $A \in \mathcal{A}$ and every value $v \in V_A$, there is an object $o \in O$ such that $o_A = v$. A presentation of O is defined as a choice of a subset of attributes and a ranking of them.

Definition 1. *A **presentation** of the object set $O \subset \prod_{A\in\mathcal{A}} V_A$ is a vector $\widehat{A} \equiv (A_1, \cdots, A_K)$ where $A_k \in \mathcal{A}$ for all $k = 1\cdots, K$ and, for every pair of objects $o, o' \in O$, there is an attribute A_k such that $o_{A_k} \neq o'_{A_k}$.*

One may note that, a natural richness condition is imposed, requiring that the chosen set of attributes be sufficiently informative so that each object can be uniquely identified.

Fixing an arbitrary presentation $\widehat{A} = (A_k)_{k=1}^K$, we introduce several notions. First, for an object $o \in O$ and an integer $k = 1, \cdots, K$, the vector of first k attribute values $(o_{A_1}, \cdots, o_{A_k})$ is called a sub-presentation of o. Second, fixing a $k = 1, \cdots, K$, $O^k \equiv \{(o_{A_1}, \cdots, o_{A_k}) : o \in O\}$ denotes the set of all feasible sub-presentations of length k. Evidently, $O^K = O$. Third, for any $k = 2, \cdots, K$ and feasible sub-presentation $\widehat{v} \equiv (v_1, \cdots, v_{k-1}) \in O^{k-1}$, $O^k|\widehat{v} \equiv \{v \in V_k : (\widehat{v}, v) \in O^k\}$ denotes the set of conditionally feasible values of the k-th attribute. To simplify notation, let $O^0 = \{\emptyset\}$ and $O^1|\emptyset \equiv O^1$.

Definition 2. *Given a presentation $\widehat{A} = (A_k)_{k=1}^K$ of $O \subset \prod_{A\in\mathcal{A}} V_A$, a preference $P_0 \in \mathcal{P}$ is **lexicographic** if, $\forall k = 1, \cdots, K$ and sub-presentation $\widehat{v} \in O^{k-1}$, there is a strict preference on $O^k|\widehat{v}$, denoted $P_0^k|\widehat{v}$, such that $o\ P_0\ o'$ iff there exists $k = 1, \cdots, K$ such that $(o_{A_1}, \cdots, o_{A_{k-1}}) = (o'_{A_1}, \cdots, o'_{A_{k-1}})$ and $o_{A_k}\ P_0^k|(o_{A_1}, \cdots, o_{A_{k-1}})\ o'_{A_k}$. Moreover, we call the set of such preferences the **domain induced by** $\widehat{A}$ and denote it $\mathcal{D}_{\widehat{A}}$.*

In the above definition, the preferences on marginally feasible attribute values, i.e., $P_0^k|\widehat{v}$'s, are called the marginal preferences. One notable feature of our definition is that, for different sub-presentations $\widehat{v}, \widehat{v}' \in O^{k-1}$, even if the corresponding sets of conditionally feasible attribute values are the same, i.e., $O^k|\widehat{v} = O^k|\widehat{v}'$, the marginal preferences are allowed to deffer from each other, i.e., $P_0^k|\widehat{v}$ can be different from $P_0^k|\widehat{v}'$. A specific example is below, where the notions defined above are illustrated with specific values.

Example 1. *Consider five flats below, where m and c refer to "modern" and "contemporary" furnishing styles. Floor area is in square meters.*

	f^1	f^2	f^3	f^4	f^5
No. of Bedrooms (NBe)	2	2	3	3	3
No. of Bathrooms (NBa)	1	2	1	2	3
Furnishing Style (FS)	m	c	m	c	c
Floor Area (FA)	55	65	85	95	110

Let $\mathcal{A} = \{NBe, NBa, FS, FA\}$ denote the set of attributes. Then the attribute values are respectively $V_{NBe} = \{2,3\}$, $V_{NBa} = \{1,2,3\}$, $V_{FS} = \{m,c\}$, and $V_{FA} = \{55,65,85,95,110\}$. Apparently, the object set is a proper subset of the product $\prod_{A\in\mathcal{A}} V_A$.

First, it's evident that (NBe, FS) is not a presentation as f^4 and f^5 can not be differentiated. Below are three different presentations and the induced preference domains.

- *The first presentation we consider is $\widehat{A}^1 \equiv (FA)$, where every flat is presented by revealing only its floor area. It is evident that the induced domain is unrestricted, i.e., $\mathcal{D}_{\widehat{A}^1} = \mathcal{P}$.*

- *The second presentation we consider is $\widehat{A}^2 \equiv (NBa, NBe)$, where every flat is presented as a sequence of its number of bathrooms and number of bedrooms. In this case, the flats are presented as $f^1 = (1,2)$, $f^2 = (2,2)$, $f^3 = (1,3)$, $f^4 = (2,3)$, and $f^5 = (3,3)$. The induced domain $\mathcal{D}_{\widehat{A}^2}$ is the set of preferences below.*

$$\begin{array}{rl|rl}
P_1: & f^1 \succ f^3 \succ f^2 \succ f^4 \succ f^5 & P_5: & f^1 \succ f^3 \succ f^5 \succ f^2 \succ f^4 \\
P_2: & f^1 \succ f^3 \succ f^4 \succ f^2 \succ f^5 & P_6: & f^1 \succ f^3 \succ f^5 \succ f^4 \succ f^2 \\
P_3: & f^3 \succ f^1 \succ f^2 \succ f^4 \succ f^5 & P_7: & f^3 \succ f^1 \succ f^5 \succ f^2 \succ f^4 \\
P_4: & f^3 \succ f^1 \succ f^4 \succ f^2 \succ f^5 & P_8: & f^3 \succ f^1 \succ f^5 \succ f^4 \succ f^2 \\
 & & & \\
P_9: & f^2 \succ f^4 \succ f^1 \succ f^3 \succ f^5 & P_{13}: & f^2 \succ f^4 \succ f^5 \succ f^1 \succ f^3 \\
P_{10}: & f^2 \succ f^4 \succ f^3 \succ f^1 \succ f^5 & P_{14}: & f^2 \succ f^4 \succ f^5 \succ f^3 \succ f^1 \\
P_{11}: & f^4 \succ f^2 \succ f^1 \succ f^3 \succ f^5 & P_{15}: & f^4 \succ f^2 \succ f^5 \succ f^1 \succ f^3 \\
P_{12}: & f^4 \succ f^2 \succ f^3 \succ f^1 \succ f^5 & P_{16}: & f^4 \succ f^2 \succ f^5 \succ f^3 \succ f^1 \\
 & & & \\
P_{17}: & f^5 \succ f^1 \succ f^3 \succ f^2 \succ f^4 & P_{21}: & f^5 \succ f^2 \succ f^4 \succ f^1 \succ f^3 \\
P_{18}: & f^5 \succ f^1 \succ f^3 \succ f^4 \succ f^2 & P_{22}: & f^5 \succ f^4 \succ f^2 \succ f^1 \succ f^3 \\
P_{19}: & f^5 \succ f^3 \succ f^1 \succ f^2 \succ f^4 & P_{23}: & f^5 \succ f^2 \succ f^4 \succ f^3 \succ f^1 \\
P_{20}: & f^5 \succ f^3 \succ f^1 \succ f^4 \succ f^2 & P_{24}: & f^5 \succ f^4 \succ f^2 \succ f^3 \succ f^1
\end{array}$$

For illustration, we show that P_2 is lexicographic. To see this, let $F \equiv \{f^1, f^2, f^3, f^4, f^5\}$. Hence $F^1|\emptyset = \{1,2,3\}$, $F^2|1 = F^2|2 = \{2,3\}$, and $F^2|3 = \{3\}$. Then P_2 is lexicographic, defined by the following marginal preferences. Note that, although $F^2|1$ is the same as $F^2|2$, the corresponding marginal preferences are different. This difference indicates that, if there is only one bathroom in the flat, the agent prefers the two-bedroom one. If the flats have two bathrooms, the agents prefers the three-bedroom one.

$$P_2^1|\emptyset : 1 \succ 2 \succ 3 \quad P_2^2|1 : 2 \succ 3 \quad P_2^2|2 : 3 \succ 2$$

- *The last presentation we consider is $\widehat{A}^3 \equiv (NBe, FS, NBa)$, where every flat is presented as a sequence of its number of bedrooms, its furnishing style, and its number of bathrooms. Accordingly, the objects are presented as $f^1 = (2,m,1)$, $f^2 = (2,c,2)$, $f^3 = (3,m,1)$, $f^4 = (3,c,2)$, $f^5 = (3,c,3)$. In this case, the induced domain $\mathcal{D}_{\widehat{A}^3}$ is the set containing following preferences.*

$$
\begin{array}{ll|ll}
P_1: & f^1 \succ f^2 \succ f^3 \succ f^4 \succ f^5 & P_9: & f^3 \succ f^4 \succ f^5 \succ f^1 \succ f^2 \\
P_2: & f^1 \succ f^2 \succ f^3 \succ f^5 \succ f^4 & P_{10}: & f^3 \succ f^5 \succ f^4 \succ f^1 \succ f^2 \\
P_3: & f^1 \succ f^2 \succ f^4 \succ f^5 \succ f^3 & P_{11}: & f^4 \succ f^5 \succ f^3 \succ f^1 \succ f^2 \\
P_4: & f^1 \succ f^2 \succ f^5 \succ f^4 \succ f^3 & P_{12}: & f^5 \succ f^4 \succ f^3 \succ f^1 \succ f^2 \\
P_5: & f^2 \succ f^1 \succ f^3 \succ f^4 \succ f^5 & P_{13}: & f^3 \succ f^4 \succ f^5 \succ f^2 \succ f^1 \\
P_6: & f^2 \succ f^1 \succ f^3 \succ f^5 \succ f^4 & P_{14}: & f^3 \succ f^5 \succ f^4 \succ f^2 \succ f^1 \\
P_7: & f^2 \succ f^1 \succ f^4 \succ f^5 \succ f^3 & P_{15}: & f^4 \succ f^5 \succ f^3 \succ f^2 \succ f^1 \\
P_8: & f^2 \succ f^1 \succ f^5 \succ f^4 \succ f^3 & P_{16}: & f^5 \succ f^4 \succ f^3 \succ f^2 \succ f^1
\end{array}
$$

□

Remark 1. *From Example 1 above, one may note that the lexicographic preferences can be equivalently defined as follows. Given a presentation $\widehat{A} = (A_k)_{k=1}^K$ of $O \subset \prod_{A \in \mathcal{A}} V_A$, a preference $P_0 \in \mathcal{P}$ is lexicographic if and only if, $\forall k = 1, \cdots, K$ and any sub-presentation $\hat{v} \in O^k$, the objects in the set $\{o \in O : (o_{A_1}, \cdots, o_{A_k}) = \hat{v}\}$ are adjacent.* □

3 Results

Among all presentations, we are interested in the ones which induce preference domains allowing for designing a good random allocation rule. It turns out that, these presentations share a common feature, as defined below.

Definition 3. *A presentation is called* ***binary*** *if every set of conditionally feasible attribute values contains at most two elements.*

For instance, the presentation $\widehat{A}^3$ in Example 1 is binary while $\widehat{A}^1$ and $\widehat{A}^2$ are not. It's worth noting that, attributes in a binary presentation can be non-binary in nature. What's required is that all attributes are *conditionally* binary. To see this, one may note that the attribute NBa is not binary in nature. However, it is conditionally binary in presentation $\widehat{A}^3$.

Theorem 1. *A presentation induces a domain which admits a strategy-proof, efficient, and envy-free rule if and only if it is binary.*

Proof. To see sufficiency, recall that Lemma 1 and 2 together imply that the PS rule satisfies all three axioms on any SDD. Hence, it suffices to show that the domain induced by any binary presentation is a subset of some SDD, which is stated as the lemma below and the proof is in Appendix A.

Lemma 3. *The preference domain induced by any binary presentation is a subset of some sequentially dichotomous domain.*

For necessity, we introduce a new notion called the block elevating property. This is a pattern that a preference domain may exhibit.[5]

[5]The block elevating property defined here can be seen as a generalization of the elevating property by Liu and Zeng (2019), where all three blocks are required to be singletons.

Definition 4. *A domain $\mathcal{D}$ exhibits the* ***block elevating property*** *if there are three preferences $\bar{P}_0, P_0, \hat{P}_0 \in \mathcal{D}$, three nonempty blocks $O_1, O_2, O_3 \subset O$ and two blocks $\underline{O}, \overline{O} \subset O$ such that $O_1 \cup O_2 \cup O_3 \cup \underline{O} \cup \overline{O} = O$ and*

$$\begin{array}{ll} \bar{P}_0: & \underline{O} \succ O_1 \succ O_3 \succ O_2 \succ \overline{O} \\ P_0: & \underline{O} \succ O_1 \succ O_2 \succ O_3 \succ \overline{O} \\ \hat{P}_0: & \underline{O} \succ O_2 \succ O_1 \succ O_3 \succ \overline{O} \end{array}$$

The necessity of Theorem 1 is implied by the following two lemmas. In particular, Lemma 4 states that, the preference domain induced by any non-binary presentation exhibits the block elevating property. Lemma 5 then states that there is no strategy-proof, efficient, and envy-free rule on any of such domains.

Lemma 4. *If a presentation is not binary, its induced domain exhibits block elevating property.*

Lemma 5. *On any preference domain exhibiting block elevating property, there exists no rule satisfying strategy-proofness, efficiency, and envy-freeness.*

The proof of Lemma 5 is in Appendix B and we explain the verification of Lemma 4 here. By definition, a presentation being non-binary means that, for some sub-presentation $v \in O^{k-1}$, there are at least three conditionally feasible values of the k-th attribute, i.e., $|O^k|v| \geqslant 3$. Pick any three values $v_1, v_2, v_3 \in O^k|v$ and let $O_t \equiv \{o \in O : (o_{A_1}, \cdots, o_{A_k}) = v_t\}$ for $t = 1, 2, 3$. Referring to Remark 1, it's evident that there exist three preferences like the ones in the definition of block elevating property. For specific illustration, one may refer to the presentation $\widehat{A}$ in Example 1, where P_1, P_5, and P_9 exhibit the structure wanted. $\square$

Knowing Theorem 1, we are hence interested in the following question: If envy-freeness is weakened to equal treatment of equals, is it possible to find more admissible presentations? The theorem below gives an answer in the negative, in the help of two technical conditions on problem size.

Theorem 2. *Suppose that the problem size satisfies Condition 1 and 2. A presentation induces a domain which admits a strategy-proof, efficient, and equal-treatment-of-equals rule if and only if it is binary.*

Proof. Since envy-freeness implies equal treatment of equals, the sufficiency part of Theorem 1 implies the sufficiency here.

The necessity is implied by Lemma 4 and the lemma below, for which Condition 1 and 2 are in Appendix C and the proof is in Appendix D.

Lemma 6. *On any preference domain exhibiting block elevating property, if the problem size satisfies Condition 1 and 2, then there exists no rule satisfying jointly strategy-proofness, efficiency, and equal treatment of equals.*

$\square$

Remark 2. *A lesson can be drawn from existing studies on random allocation is that, when envy-freeness is weakened to equal treatment of equals, the characterization of admissible rules (including impossibilities) becomes much more difficult. This can also be seen from the proof* *of Lemma 6, which turns out much more complicated than that of Lemma 5. Our proof rely on Condition 1 and 2, which have been verified numerically for all problem sizes no larger than 3000. Moreover, we managed to identify the smallest problem size with which Condition 1 fails: 13799. For specific applications, one can check easily by computer whether these conditions hold. In what follows, we explain briefly the role of Condition 1 and 2 in the proof. To do this, let $m_1 \equiv |O_1|$ and $m_2 \equiv |O_2|$, where O_1 and O_2 are subsets of objects in the definition of block elevating property. The proof of Lemma 6 is by contradiction, meaning that we assume the existence of a rule satisfying the axioms in the lemma and then identify a contradiction. For the cases where $\frac{m_1 n}{m_1+m_2}$ is an integer, the contradiction is found without the help of conditions on n. While, for the cases where $\frac{m_1 n}{m_1+m_2}$ is not an integer, we need the conditions on n to characterize the random allocations and hence the contradiction.* $\square$

Remark 3. *Bogomolnaia and Moulin (2001) presents an impossibility that there is no rule satisfying strategy-proofness, efficiency, and equal treatment of equals. The essential part of the proof there concerns the situation where $n = 4$. One may hence suspect that the proof strategy there can be used to prove, in the current setting, that there is no rule satisfying the aforementioned axioms when the presentation involves a set of conditionally feasible values that contains at least four elements. It seems that the only difference is that the preferences in Bogomolnaia and Moulin (2001) are over four objects and here it's over four blocks of objects. Unfortunately, the proof strategy there does not work in the current setting. The key issue is that the implications of efficiency become less precise. In particular, for the situation where preferences are over objects, whenever there is a preference reversal in agents' preferences, efficiency implies precisely which agent gets no probability of which object. When the preference reversal is between two blocks of objects, such implication is unclear and depending on the size of blocks.*

4 Conclusion

In this paper, we propose a model studying how to present the objects for random allocation in a way that the induced preferences of agents' allow for designing a good allocation rule. Our modeling of how a presentation affects preference formation is inspired by observations in reality and our results support practices where objects are presented by sequentially revealing attributes that are conditionally binary. In particular, we show that such presentations are the unique ones which induce preferences allowing for strategy-proof, efficient, and envy-free allocation rules. When envy-freeness is weaken to equal treatment of equals, we are able to prove the characterization only when two technical conditions on problem size are satisfied. Hence, a question for further exploration is whether the characterization still holds without the help of these conditions. Another interesting question is, given such a binary presentation, whether the probabilistic serial rule is uniquely desirable.

Appendix

A Proof of Lemma 3

We present first the definition of sequentially dichotomous domains. To do so, we need some preliminary notions. First, given the object set O, a partition of it, denoted $\mathcal{O}$, is a set of nonempty subsets of O such that every object is contained in exactly one of these subsets. We call these subsets blocks. Next, a partition $\mathcal{O}'$ is a dichotomous refinement of another partition $\mathcal{O}$ if exactly one block in the latter breaks into two smaller blocks in the former and all others remain unchanged. Finally, a sequence of partitions, denoted $(\mathcal{O}_t)_{t=1}^n$, is called a dichotomous path, if (i) $\mathcal{O}_1 = \{O\}$, (ii) $\mathcal{O}_n = \{\{o\} : o \in O\}$, and (iii) $\mathcal{O}_t$ is a dichotomous refinement of $\mathcal{O}_{t-1}$ for all $t = 2, \cdots, n$. Put otherwise, a dichotomous path breaks the grand block O into singletons by finer and finer dichotomous refinements. We say a preference $P_0 \in \mathcal{P}$ respects a partition $\mathcal{O}$ if, for every block, the contained objects are adjacent in P_0. Further, we say a preference respects a dichotomous path, if it respects every partition in it. An SDD is hence defined as the set of preferences that respect a particular dichotomous path. For detailed discussion and examples of SDD, please refer to Liu (2020).

By Remark 1, to see that the preference domain induced by a binary presentation is a subset of an SDD, it suffices to note that, by sequentially dividing groups of objects by conditionally feasible attribute values according to the given presentation, one can easily construct the dichotomous path and note that every induced preference respects that path.

B Proof of Lemma 5

We prove the lemma by contradiction. Let $\mathcal{D}$ be a domain exhibiting block elevating property and $\varphi : \mathcal{D}^n \to \mathcal{L}$ a rule satisfying all three axioms in the lemma. Given a random assignment L and a subset of objects $O' \subset O$, we denote $L_{iO'} \equiv \sum_{x \in O'} L_{ix}$. By definition of block elevating property, the following two preference profiles, denoted P and P', are admissible. In particular, agents 1 to $n-1$ have the same preference and agent 1 is the unilateral deviator whose preference changes from P_1 in P to P'_1 in P'.

$$
\begin{array}{rl}
P_1: & \underline{O} \succ O_1 \succ O_2 \succ O_3 \succ \overline{O} \\
P_2: & \underline{O} \succ O_1 \succ O_2 \succ O_3 \succ \overline{O} \\
\vdots & \vdots \\
P_{n-1}: & \underline{O} \succ O_1 \succ O_2 \succ O_3 \succ \overline{O} \\
P_n: & \underline{O} \succ O_1 \succ O_3 \succ O_2 \succ \overline{O}
\end{array}
\qquad\qquad
\begin{array}{rl}
P'_1: & \underline{O} \succ O_2 \succ O_1 \succ O_3 \succ \overline{O} \\
P_2: & \underline{O} \succ O_1 \succ O_2 \succ O_3 \succ \overline{O} \\
\vdots & \vdots \\
P_{n-1}: & \underline{O} \succ O_1 \succ O_2 \succ O_3 \succ \overline{O} \\
P_n: & \underline{O} \succ O_1 \succ O_3 \succ O_2 \succ \overline{O}
\end{array}
$$

By the assumption that the rule at hand satisfies the aforementioned axioms, we pin down the relevant probabilities, as follows.

$\varphi(P)$:	O_1	O_2
1	$\frac{\lvert O_1\rvert}{n}$	$\frac{\lvert O_2\rvert}{n-1}$
2	$\frac{\lvert O_1\rvert}{n}$	$\frac{\lvert O_2\rvert}{n-1}$
$\vdots$	$\vdots$	
$n-1$	$\frac{\lvert O_1\rvert}{n}$	$\frac{\lvert O_2\rvert}{n-1}$
n	$\frac{\lvert O_1\rvert}{n}$	0

$\varphi(P')$:	O_1	O_2
1	0	$\frac{\lvert O_2\rvert}{n-1} + \frac{(n-2)\lvert O_1\rvert}{(n-1)^2}$
2	$\frac{\lvert O_1\rvert}{n-1}$	$\frac{\lvert O_2\rvert}{n-1} - \frac{\lvert O_1\rvert}{(n-1)^2}$
$\vdots$	$\vdots$	
$n-1$	$\frac{\lvert O_1\rvert}{n-1}$	$\frac{\lvert O_2\rvert}{n-1} - \frac{\lvert O_1\rvert}{(n-1)^2}$
n	$\frac{\lvert O_1\rvert}{n-1}$	0

For $\varphi(P)$, note first that envy-freeness implies that agents receive the same probability to get O_1. Next, efficiency implies $\varphi_{nO_2}(P) = 0$. Suppose otherwise, efficiency requires $\varphi_{iO_3}(P) = 0$ for all agents $i = 1, \cdots, n-1$. Then feasibility implies $\varphi_{nO_3}(P) = |O_3| \geqslant 1$, which means agent n has a probability larger than one to get some object: a contradiction to the definition of random allocations. Finally, envy-freeness requires that agents 1 to $n-1$ receive the same probability to get O_2, i.e., $\varphi_{iO_2} = \frac{|O_2|}{n-1}$ for all $i = 1, \cdots, n-1$.

For $\varphi(P')$, note first that efficiency implies $\varphi_{1O_1}(P') = 0$. To see this, suppose $\varphi_{1O_1}(P') > 0$. Then efficiency requires $\varphi_{iO_2}(P') = 0$ for all $i = 2, \cdots, n$ and hence feasibility implies $\varphi_{1O_2}(P') = |O_2| \geqslant 1$: impossible. Second, envy-freeness implies that agents 2 to n receive the same probability to get O_1, i.e., $\varphi_{iO_1}(P') = \frac{|O_1|}{n-1}$. Third, by similar argument as that in the first step, we have $\varphi_{nO_2}(P') = 0$. Last, by envy-freeness, we have $\varphi_{1O_3}(P') = \varphi_{iO_3}(P')$ for all $i = 1, \cdots, n-1$ and $\varphi_{2O_3}(P') = \varphi_{jO_3}(P')$ for all $i = 2, \cdots, n-1$. Let $x \equiv \varphi_{2O_2}(P')$. Then envy-freeness and feasibility imply the following and hence the remaining probabilities we wanted.

$$\begin{aligned}
&\varphi_{1O_1}(P') + \varphi_{1O_2}(P') + \varphi_{1O_3}(P') = \varphi_{2O_1}(P') + \varphi_{2O_2}(P') + \varphi_{2O_3}(P') \\
\Rightarrow & 0 + \varphi_{1O_3}(P') = \frac{|O_1|}{n-1} + x \\
\Rightarrow & |O_2| - (n-2)x = \frac{|O_1|}{n-1} + x \\
\Rightarrow & x = \frac{|O_2|}{n-1} - \frac{|O_1|}{(n-1)^2}
\end{aligned}$$

Given $\varphi(P)$ and $\varphi(P')$, we identify a contradiction to strategy-proofness:

$$[\varphi_{1O_1}(P)+\varphi_{1O_2}(P)]-[\varphi_{1O_1}(P')+\varphi_{1O_2}(P')] = [\frac{|O_1|}{n}+\frac{|O_2|}{n-1}]-[0+\frac{|O_2|}{n-1}+\frac{(n-2)|O_1|}{(n-1)^2}] = \frac{|O_1|}{n(n-1)^2} > 0.$$

C Two Technical Conditions

Condition 1.

$$\begin{aligned}
f(m_1, m_2) \equiv & -n(m_1+m_2)\left(\left[\frac{m_2 n}{m_1+m_2}\right]\right)^2 \\
& + \left[(n^2-n+1)m_1 + (2n^2-n)m_2\right]\left(\left[\frac{m_2 n}{m_1+m_2}\right]\right) - n^2(n-1)m_2 \leqslant 0
\end{aligned}$$

for all positive integers m_1, m_2 such that $m_1 + m_2 < n$ and $\frac{m_2 n}{m_1+m_2}$ is not an integer, where $[x]$ denotes for a real number the largest integer which is no greater than x.

Condition 2.

$$g(m_1, m_2, m_3) \equiv \frac{m_3 - 2(\frac{m}{n} - \frac{m_1}{n-(\bar{n}_5-1)}) - (\bar{n}_5 - 1)\times\gamma(\bar{n}_5)}{n-(\bar{n}_5+1)} \neq \frac{m_3}{n}$$

for all positive integers m_1, m_2, m_3 such that $m_1 + m_2 + m_3 \leqslant n$ and $\frac{m_2 n}{m_1+m_2}$ is not an integer.

$$\textit{where} \quad \bar{n}_5 = \left[\frac{m_2 n}{m_1 + m_2}\right] + 1,$$

$$\gamma(k) = \frac{\Gamma_1(k)m_1 + \Gamma_2(k)m_2 + \Gamma_3(k)m_3}{n[n^4 - 2(k+1)n^3 + (k^2+5k-1)n^2 - (3k^2+k-2)n + 2(k^2-k)]}$$

$$\textit{and} \quad \Gamma_1(k) = 2(k-2)n^2 - 2(k^2-k-2)n + 2(k^2-k-2)$$

$$\Gamma_2(k) = -2n^3 + 4kn^2 - 2(k^2+k-1)n + 2(k^2-k)$$

$$\Gamma_3(k) = n^4 - 2(k+1)n^3 + (k^2+5k-1)n^2 - (3k^2+k-2)n + 2(k^2-k)$$

D Proof of Lemma 6

Let $m_1 \equiv |O_1| \geqslant 1$, $m_2 \equiv |O_2| \geqslant 1$, $m_3 \equiv |O_3| \geqslant 1$, and $m \equiv m_1 + m_2 + m_3$. For a real number x, $[x]$ denotes the largest integer which is no larger than x. Finally given a random assignment L and a subset of objects $O' \subset O$, let $L_{iO'} \equiv \sum_{x \in O'} L_{ix}$. For convenience, we call strategy-proofness 'SP', efficiency 'EFF', and equal treatment of equals 'ETE'.

Let $\mathcal{D} \equiv \{\bar{P}_0, P_0, \hat{P}_0\}$ where the preferences are the ones in the definition of bloc elevating property. To prove the lemma, it suffices to prove that $\mathcal{D}$ admits no rule satisfying the aforementioned axioms. Suppose not, and let $\varphi : \mathcal{D}^n \longrightarrow \mathcal{L}$ be a rule satisfying these axioms.

The first observation we have is as follows, which can be proved by applying SP and ETE. The proof is standard and hence omitted.

Lemma 7. *For any $P \in \mathcal{D}^n$, $\varphi_{iO_1}(P) + \varphi_{iO_2}(P) + \varphi_{iO_3}(P) = \frac{m}{n}$ for all $i \in I$.*

Since $\frac{m_1 n}{m_1+m_2} + \frac{m_2 n}{m_1+m_2} = n$, it's either both $\frac{m_1 n}{m_1+m_2}$ and $\frac{m_2 n}{m_1+m_2}$ are integers or neither one of them is an integer. One shall note that, only for the case where they are not integers do we need Condition 1 and 2.

In order to identify contradictions, we will construct six groups of profiles and characterize for each of them the probabilities associated to O_1, O_2, O_3. The contradiction for the case where $\frac{m_2 n}{m_1+m_2}$ is an integer is found using only the first four groups of profiles. While for the case where $\frac{m_2 n}{m_1+m_2}$ is not an integer, we need in addition the remaining two groups of profiles.

We first list all profiles groups, as follows.

Profile group 1:

$$\begin{aligned}
P^{1,0} &= (P_1, \cdots, P_n) \\
P^{1,1} &= (\hat{P}_1, P_2, \cdots, P_n) \\
&\vdots \\
P^{1,k} &= (\hat{P}_1, \cdots, \hat{P}_k, P_{k+1}, \cdots, P_n) \\
&\vdots \\
P^{1,\bar{n}_1} &= (\hat{P}_1, \cdots, \hat{P}_{\bar{n}_1}, P_{\bar{n}_1+1}, \cdots, P_n)
\end{aligned}$$

$$\text{where } \bar{n}_1 = \begin{cases} \frac{m_2 n}{m_1+m_2}, & \text{if } \frac{m_2 n}{m_1+m_2} \text{ is integer} \\ \left[\frac{m_2 n}{m_1+m_2}\right], & \text{otherwise} \end{cases}$$

Profile group 2:

$$\begin{aligned}
P^{2,1} &= (P_1, \cdots, P_{n-1}, \bar{P}_n) \\
P^{2,2} &= (\hat{P}_1, P_2 \cdots, P_{n-1}, \bar{P}_n) \\
&\vdots \\
P^{2,k} &= (\hat{P}_1, \cdots, \hat{P}_{k-1}, P_k, \cdots, P_{n-1}, \bar{P}_n) \\
&\vdots \\
P^{2,\bar{n}_2} &= (\hat{P}_1, \cdots, \hat{P}_{\bar{n}_2-1}, P_{\bar{n}_2}, \cdots, P_{n-1}, \bar{P}_n)
\end{aligned}$$

$$\text{where } \bar{n}_2 = \begin{cases} \frac{m_2 n}{m_1+m_2}, & \text{if } \frac{m_2 n}{m_1+m_2} \text{ is integer} \\ \left[\frac{m_2 n}{m_1+m_2}\right] + 1, & \text{otherwise} \end{cases}$$

Profile group 3:

$$
\begin{aligned}
P^{3,0} &= (\hat{P}_1, \cdots, \hat{P}_n) \\
P^{3,1} &= (\hat{P}_1, \cdots, \hat{P}_{n-1}, P_n) \\
&\vdots \\
P^{3,k} &= (\hat{P}_1, \cdots, \hat{P}_{n-k}, P_{n-k+1}, \cdots, P_n) \\
&\vdots \\
P^{3,\bar{n}_3} &= (\hat{P}_1, \cdots, \hat{P}_{n-\bar{n}_3}, P_{n-\bar{n}_3+1}, \cdots, P_n)
\end{aligned}
$$

where $\bar{n}_3 = \begin{cases} \frac{m_1 n}{m_1+m_2}, & \text{if } \frac{m_2 n}{m_1+m_2} \text{ is integer} \\ \left[\frac{m_1 n}{m_1+m_2}\right], & \text{otherwise} \end{cases}$

Profile group 4:

$$
\begin{aligned}
P^{4,1} &= (\hat{P}_1, \cdots, \hat{P}_{n-1}, \bar{P}_n) \\
P^{4,2} &= (\hat{P}_1, \cdots, \hat{P}_{n-2}, P_{n-1}, \bar{P}_n) \\
&\vdots \\
P^{4,k} &= (\hat{P}_1, \cdots, \hat{P}_{n-k}, P_{n-k+1}, \cdots, P_{n-1}, \bar{P}_n) \\
&\vdots \\
P^{4,\bar{n}_4} &= (\hat{P}_1, \cdots, \hat{P}_{n-\bar{n}_4}, P_{n-\bar{n}_4+1}, \cdots, P_{n-1}, \bar{P}_n)
\end{aligned}
$$

where $\bar{n}_4 = \begin{cases} \frac{m_1 n}{m_1+m_2}, & \text{if } \frac{m_2 n}{m_1+m_2} \text{ is integer} \\ \left[\frac{m_1 n}{m_1+m_2}\right], & \text{otherwise} \end{cases}$

Profile group 5:

$$
\begin{aligned}
P^{5,1} &= (P_1, \cdots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n) \\
P^{5,2} &= (\hat{P}_1, P_2 \cdots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n) \\
&\vdots \\
P^{5,k} &= (\hat{P}_1, \cdots, \hat{P}_{k-1}, P_k, \cdots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n) \\
&\vdots \\
P^{5,\bar{n}_5} &= (\hat{P}_1, \cdots, \hat{P}_{\bar{n}_5-1}, P_{\bar{n}_5}, \cdots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)
\end{aligned}
$$

$\bar{n}_5 = \left[\frac{m_2 n}{m_1+m_2}\right] + 1$ and $\frac{m_2 n}{m_1+m_2}$ is not integer.

Profile group 6:

$$
\begin{aligned}
P^{6,1} &= (\hat{P}_1, \cdots, \hat{P}_{n-2}, P_{n-1}, \bar{P}_n) \\
P^{6,2} &= (\hat{P}_1, \cdots, \hat{P}_{n-2}, \bar{P}_{n-1}, \bar{P}_n) \\
P^{6,3} &= (\hat{P}_1, \cdots, \hat{P}_{n-3}, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n) \\
&\vdots \\
P^{6,k} &= (\hat{P}_1, \cdots, \hat{P}_{n-k}, P_{n-k+1}, \cdots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n) \\
&\vdots \\
P^{6,\bar{n}_6} &= (\hat{P}_1, \cdots, \hat{P}_{n-\bar{n}_6}, P_{n-\bar{n}_6+1}, \cdots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)
\end{aligned}
$$

$\bar{n}_6 = \left[\frac{m_1 n}{m_1+m_2}\right]$ and $\frac{m_2 n}{m_1+m_2}$ is not integer.

We then characterize the random allocations for the preference profiles listed above, by applying the axioms mentioned in Lemma 6.

Claim 1. *For each preference profile $P^{1,k}$, $\varphi(P^{1,k})$ specifies probabilities on O_1, O_2, and O_3 as follows*

$$
\begin{array}{cccc}
 & O_1 & O_2 & O_3 \\
1 & 0 & \frac{m_1+m_2}{n} & \frac{m_3}{n} \\
\vdots & \vdots & \vdots & \vdots \\
k & 0 & \frac{m_1+m_2}{n} & \frac{m_3}{n} \\
k+1 & \frac{m_1}{n-k} & \frac{m_1+m_2}{n} - \frac{m_1}{n-k} & \frac{m_3}{n} \\
\vdots & \vdots & \vdots & \vdots \\
n & \frac{m_1}{n-k} & \frac{m_1+m_2}{n} - \frac{m_1}{n-k} & \frac{m_3}{n}
\end{array}
$$

Proof. Verification of the claim consists of three steps.

Step 1: We show $\varphi_{iO_3}(P^{1,k}) = \frac{m_3}{n}$ for all $i \in I$ and all $k = 0, 1, \cdots, \bar{n}_1$. First, by ETE, $\varphi_{iO_3}(P^{1,0}) = \frac{m_3}{n}$ for all $i = 1, \cdots, n$. Second, for all $k = 1, \cdots, \bar{n}_1$, if $\varphi_{iO_3}(P^{1,k-1}) = \frac{m_3}{n}$ for all $i \in I$, then $\varphi_{iO_3}(P^{1,k}) = \frac{m_3}{n}$ for all $i \in I$. Notice that $P^{1,k}$ and $P^{1,k-1}$ are different only in agent k's preference. Then SP implies $\varphi_{kO_3}(P^{1,k}) = \varphi_{kO_3}(P^{1,k-1}) = \frac{m_3}{n}$. Hence by feasibility and ETE, $\varphi_{iO_3}(P^{1,k}) = \frac{m_3}{n}$ for all $i \in I$.
Step 2: We show $\varphi_{iO_1}(P^{1,k}) = 0$ for all $i = 1, \cdots, k$ and all $k = 0, 1, \cdots, \bar{n}_1$. Fix a k and suppose without loss of generality $\varphi_{1O_1}(P^{1,k}) = \beta > 0$. Then EFF implies $\varphi_{iO_2}(P^{1,k}) = 0$ for all $i = k+1, \cdots, n$ and ETE implies $\varphi_{iO_2}(P^{1,k}) = \frac{m_2}{k}$ for all $i = 1, \cdots, k$. Then we have below a contradiction to Lemma 7.

$$\varphi_{1O_1}(P^{1,k}) + \varphi_{1O_2}(P^{1,k}) + \varphi_{1O_3}(P^{1,k}) = \beta + \frac{m_2}{k} + \frac{m_3}{n} > \frac{m_2}{k} + \frac{m_3}{n} \geqslant \frac{m}{n}$$

where the last inequality comes from $k \leqslant \bar{n}_1 \leqslant \frac{m_2 n}{m_1+m_2}$.
Step 3: Lemma 7 and ETE imply all other probabilities we wanted. $\square$

Claim 2. *For each preference profile $P^{2,k}$, $\varphi(P^{2,k})$ specifies probabilities on O_1, O_2, and O_3 as follows*

$$\begin{array}{cccc}
 & O_1 & O_2 & O_3 \\
1 & 0 & \frac{m}{n} - \alpha(k) & \alpha(k) \\
\vdots & \vdots & \vdots & \vdots \\
k-1 & 0 & \frac{m}{n} - \alpha(k) & \alpha(k) \\
k & \frac{m_1}{n-(k-1)} & \frac{m_2-(k-1)\times(\frac{m}{n}-\alpha(k))}{n-k} & \frac{m_3-(\frac{m}{n}-\frac{m_1}{n-(k-1)})-(k-1)\times\alpha(k)}{n-k} \\
\vdots & \vdots & \vdots & \vdots \\
n-1 & \frac{m_1}{n-(k-1)} & \frac{m_2-(k-1)\times(\frac{m}{n}-\alpha(k))}{n-k} & \frac{m_3-(\frac{m}{n}-\frac{m_1}{n-(k-1)})-(k-1)\times\alpha(k)}{n-k} \\
n & \frac{m_1}{n-(k-1)} & 0 & \frac{m}{n} - \frac{m_1}{n-(k-1)}
\end{array}$$

where $\alpha(k) = \frac{(k-2)m_1-(n-(k-1))m_2+(n-1)(n-(k-1))m_3}{n(n-1)(n-(k-1))}$.

Proof. Verification of the claim consists of six steps.
Step 1: We show $\varphi(P^{2,1})$ specifies probabilities on O_1, O_2, and O_3 as follows

$$\begin{array}{cccc}
 & O_1 & O_2 & O_3 \\
1 & \frac{m_1}{n} & \frac{m_2}{n-1} & \frac{m_2+m_3}{n} - \frac{m_2}{n-1} \\
\vdots & \vdots & \vdots & \vdots \\
n-1 & \frac{m_1}{n} & \frac{m_2}{n-1} & \frac{m_2+m_3}{n} - \frac{m_2}{n-1} \\
n & \frac{m_1}{n} & 0 & \frac{m_2+m_3}{n}
\end{array}$$

First notice that $P^{2,1}$ and $P^{1,0}$ are different only in agent n's preference. Then SP implies $\varphi_{nO_1}(P^{2,1}) = \varphi_{nO_1}(P^{1,0}) = \frac{m_1}{n}$. Hence feasibility and ETE imply $\varphi_{iO_1}(P^{2,1}) = \frac{m_1}{n}$ for all $i \in I$. Second $\varphi_{nO_2}(P^{2,1}) = 0$. Suppose not, then EFF implies $\varphi_{iO_3}(P^{2,1}) = 0$ for all $i = 1, \cdots, n-1$. Hence feasibility implies $\varphi_{nO_3}(P^{2,1}) = m_3 \geqslant 1$: a contradiction against Lemma 7. Last, feasibility and ETE imply all other probabilities we wanted.

Step 2: We show $\varphi_{nO_1}(P^{2,k}) = \frac{m_1}{n-(k-1)}$ for all $k = 2, \cdots, \bar{n}_2$. Fix a k. Notice that $P^{2,k}$ and $P^{1,k-1}$ are different only in agent n's preference. Then SP implies $\varphi_{nO_1}(P^{2,k}) = \varphi_{nO_1}(P^{1,k-1}) = \frac{m_1}{n-(k-1)}$.

Step 3: We show $\varphi_{nO_2}(P^{2,k}) = 0$ for all $k = 2, \cdots, \bar{n}_2$. Fix a k and suppose $\varphi_{nO_2}(P^{2,k}) > 0$. Then EFF implies $\varphi_{iO_3}P^{2,k} = 0$ for all $i = 1, \cdots, n-1$ and hence $\varphi_{nO_3}P^{2,k} = m_3$: a contradiction against Lemma 7.

Step 4: We show $\varphi_{iO_3}(P^{2,k}) = \alpha(k)$ for all $i = 1, \cdots, k-1$ and all $k = 2, \cdots, \bar{n}_2$.

First we show $\varphi_{1O_3}(P^{2,2}) = \alpha(2)$. Notice that $P^{2,2}$ and $P^{2,1}$ are different only in agent 1's preference. Then SP implies $\varphi_{1O_3}(P^{2,2}) = \varphi_{1O_3}(P^{2,1}) = \frac{m_2+m_3}{n} - \frac{m_2}{n-1}$ and hence

$$\alpha(2) = \frac{(2-2)m_1 - (n-(2-1))m_2 + (n-1)(n-(2-1))m_3}{n(n-1)(n-(2-1))} = \frac{(n-1)m_3 - m_2}{n(n-1)} = \frac{m_2+m_3}{n} - \frac{m_2}{n-1}.$$

Second, we show an induction: If $\varphi_{iO_3}(P^{2,k}) = \alpha(k)$ for all $i = 1, \cdots, k-1$ and an $k \in \{2, \cdots, \bar{n}_2 - 1\}$, then $\varphi_{iO_3}(P^{2,k+1}) = \alpha(k+1)$ for all $i = 1, \cdots, k$. Notice that $P^{2,k+1}$ and $P^{2,k}$ are different only in agent k's preference. Then SP implies $\varphi_{kO_3}(P^{2,k+1}) = \varphi_{kO_3}(P^{2,k})$. Hence for all $i = 1, \cdots, k$

$$\begin{aligned}
\varphi_{iO_3}(P^{2,k+1}) &= \varphi_{kO_3}(P^{2,k+1}) && \text{by ETE} \\
&= \varphi_{kO_3}(P^{2,k}) && \text{by SP} \\
&= \tfrac{m_3 - (\frac{m}{n} - \frac{m_1}{n-(k-1)}) - (k-1)\times\varphi_{k-1 O_3}(P^{2,k})}{n-k} && \text{by feasibility and ETE} \\
&= \tfrac{m_3 - (\frac{m}{n} - \frac{m_1}{n-(k-1)}) - (k-1)\times\alpha(k)}{n-k} && \text{by induction hypothesis} \\
&= \alpha(k+1) && \text{by simplifying expression.}
\end{aligned}$$

Step 5: We show $\varphi_{iO_1}(P^{2,k}) = 0$ for all $i = 1, \cdots, k-1$ and all $k = 2, \cdots, \bar{n}_2$. Fix a k. Suppose without loss of generality $\varphi_{1O_1}(P^{2,k}) = \beta > 0$. Then ETE implies $\varphi_{iO_1}(P^{2,k}) = \beta$ for all $i = 1, \cdots, k-1$. Hence Lemma 7 and Step 4 imply $\varphi_{iO_2}(P^{2,k}) = \frac{m}{n} - \alpha(k) - \beta$ for all $i = 1, \cdots, k-1$ and EFF implies $\varphi_{iO_2}(P^{2,k}) = 0$ for all $i = k, \cdots, n-1$.

Now we show $(k-1) \times (\frac{m}{n} - \alpha(k) - \beta) < m_2$: a contradiction against feasibility.

$$\begin{aligned}
&(k-1) \times (\frac{m}{n} - \alpha(k) - \beta) < m_2 \\
\Leftarrow &(k-1) \times (\frac{m}{n} - \alpha(k)) \leqslant m_2 \\
\Leftrightarrow &(k-1) \times \left[\frac{m}{n} - \frac{(k-2)m_1 - (n-(k-1))m_2 + (n-1)(n-(k-1))m_3}{n(n-1)(n-(k-1))}\right] - m_2 \leqslant 0 \\
\Leftrightarrow &(k-1) \times [(n-1)(n-(k-1))(m_1+m_2) - (k-2)m_1 + (n-(k-1))m_2] \\
&- n(n-1)(n-(k-1))m_2 \leqslant 0 \\
\Leftrightarrow &- n(m_1+m_2)(k-1)^2 + \left[(n^2-n+1)m_1 + (2n^2-n)m_2\right](k-1) - n^2(n-1)m_2 \leqslant 0
\end{aligned}$$

Let $f(\theta) = -n(m_1+m_2)(\theta-1)^2 + [(n^2-n+1)m_1 + (2n^2-n)m_2](\theta-1) - n^2(n-1)m_2$. To verify the Step, it suffices to show $f(\theta) \leqslant 0$ for all $k = 2, \cdots, \bar{n}_2$.

From the functional form of $f(\theta)$, we have first-order derivative and the second order derivative as follows

$$\begin{aligned}
f'(\theta) &= -2n(m_1+m_2)(\theta-1) + (n^2-n+1)m_1 + (2n^2-n)m_2 \\
f''(\theta) &= -2n(m_1+m_2)
\end{aligned}$$

When $\frac{m_2 n}{m_1+m_2}$ is an integer, $\bar{n}_2 = \frac{m_2 n}{m_1+m_2}$.

$$
\begin{aligned}
f(\bar{n}_2) =& -n(m_1+m_2)\left(\frac{m_2 n}{m_1+m_2}-1\right)^2 \\
&+\left[(n^2-n+1)m_1+(2n^2-n)m_2\right]\left(\frac{m_2 n}{m_1+m_2}-1\right)-n^2(n-1)m_2 \\
=&\frac{1}{m_1+m_2}\{-n[(n-1)m_2-m_1]^2 \\
&+\left[(n^2-n+1)m_1+(2n^2-n)m_2\right][(n-1)m_2-m_1]-n^2(n-1)m_2(m_1+m_2)\} \\
=&\frac{1}{m_1+m_2}\left[-(n^2+1)m_1^2-\left((n-\frac{1}{2})^2+\frac{3}{4}\right)m_1m_2\right]<0. \\
f'(\bar{n}_2) =& -2n(m_1+m_2)(\frac{m_2 n}{m_1+m_2}-1)+(n^2-n+1)m_1+(2n^2-n)m_2 \\
=&\frac{1}{m_1+m_2}[-2n(m_1+m_2)((n-1)m_2-m_1)+(n^2-n+1)m_1(m_1+m_2) \\
&+(2n^2-n)m_2(m_1+m_2)] \\
=&\frac{1}{m_1+m_2}[(n^2+n+1)m_1^2+nm_2^2+(n^2+2n+1)m_1m_2]>0
\end{aligned}
$$

By $f''(\theta)<0$ and $f'(\bar{n}_2)>0$, $f'(\theta)>0$ for all $\theta \leqslant \bar{n}_2$, that is $f(\theta)$ is increasing through 2 to $\bar{n}_2$. Then $f(\bar{n}_2)<0$ implies $f(\theta)<0$ for all $\theta \leqslant \bar{n}_2$, which is what we want.

When $\frac{m_2 n}{m_1+m_2}$ is not an integer, $\bar{n}_2 = \left[\frac{m_2 n}{m_1+m_2}\right]+1$.

$$
\begin{aligned}
f(\bar{n}_2) =& -n(m_1+m_2)\left(\left[\frac{m_2 n}{m_1+m_2}\right]\right)^2 \\
&+\left[(n^2-n+1)m_1+(2n^2-n)m_2\right]\left(\left[\frac{m_2 n}{m_1+m_2}\right]\right)-n^2(n-1)m_2 \leqslant 0
\end{aligned}
$$

where the last inequality comes from Condition 1 in Appendix C.

$$
\begin{aligned}
f'(\bar{n}_2) =& -2n(m_1+m_2)(\frac{m_2 n}{m_1+m_2}-\delta)+(n^2-n+1)m_1+(2n^2-n)m_2 \\
=&\frac{1}{m_1+m_2}[-2n(m_1+m_2)((n-\delta)m_2-\delta m_1)+(n^2-n+1)m_1(m_1+m_2) \\
&+(2n^2-n)m_2(m_1+m_2)] \\
=&\frac{1}{m_1+m_2}[m_1 n(n-1)+m_2 n(m_1(n-2)-m_2)+m_1m_2+m_1^2 \\
&+2\delta(m_1^2 n+m_2^2 n+2m_1m_2 n)]>0
\end{aligned}
$$

where the last inequality comes from $m_2 \leqslant (n-2)$ and $m_1 \geqslant 1$.

Step 6: Lemma 7 and ETE imply all other probabilities we wanted. □

Claim 3. *For each preference profile $P^{3,k}$, $\varphi(P^{3,k})$ specifies probabilities on O_1, O_2, and O_3 as follows*

$$
\begin{array}{cccc}
 & O_1 & O_2 & O_3 \\
1 & \frac{m_1+m_2}{n} - \frac{m_2}{n-k} & \frac{m_2}{n-k} & \frac{m_3}{n} \\
\vdots & \vdots & \vdots & \vdots \\
n-k & \frac{m_1+m_2}{n} - \frac{m_2}{n-k} & \frac{m_2}{n-k} & \frac{m_3}{n} \\
n-k+1 & \frac{m_1+m_2}{n} & 0 & \frac{m_3}{n} \\
\vdots & \vdots & \vdots & \vdots \\
n & \frac{m_1+m_2}{n} & 0 & \frac{m_3}{n}
\end{array}
$$

Proof. This claim can be verified by the similar arguments that verify Claim 1. □

Claim 4. *For each preference profile $P^{4,k}$, $\varphi(P^{4,k})$ specifies probabilities on O_1, O_2, and O_3 as follows*

$$
\begin{array}{cccc}
 & O_1 & O_2 & O_3 \\
1 & \frac{m_1+m_2}{n} - \frac{m_2}{n-k} & \frac{m_2}{n-k} & \frac{m_3}{n} \\
\vdots & \vdots & \vdots & \vdots \\
n-k & \frac{m_1+m_2}{n} - \frac{m_2}{n-k} & \frac{m_2}{n-k} & \frac{m_3}{n} \\
n-k+1 & \frac{m_1+m_2}{n} & 0 & \frac{m_3}{n} \\
\vdots & \vdots & \vdots & \vdots \\
n-1 & \frac{m_1+m_2}{n} & 0 & \frac{m_3}{n} \\
n & \frac{m_1+m_2}{n} & 0 & \frac{m_3}{n}
\end{array}
$$

Proof. Verification of the claim consists of five steps.

Step 1: We show $\varphi_{nO_1}(P^{4,k}) = \frac{m_1+m_2}{n}$ for all $k = 1, \cdots, \bar{n}_4$. Fix a k. Notice that $P^{4,k}$ and $P^{3,k}$ are different only in agent n's preference. Then SP implies $\varphi_{nO_1}(P^{4,k}) = \varphi_{nO_1}(P^{3,k}) = \frac{m_1+m_2}{n}$.

Step 2: We show $\varphi_{nO_2}(P^{4,k}) = 0$ and $\varphi_{nO_3}(P^{4,k}) = \frac{m_3}{n}$ for all $k = 1, \cdots, \bar{n}_4$. Fix a k. Suppose $\varphi_{nO_2}(P^{4,k}) > 0$, then EFF implies $\varphi_{iO_3}(P^{4,k}) = 0$ for all $i = 1, \cdots, n-1$ and hence $\varphi_{nO_3}(P^{4,k}) = m_3$: a contradiction against Lemma 7. Given $\varphi_{nO_2}(P^{4,k}) = 0$, Lemma 7 implies $\varphi_{nO_3}(P^{4,k}) = \frac{m_3}{n}$.

Step 3: We show $\varphi_{iO_3}(P^{4,k}) = \frac{m_3}{n}$ for all $i = 1, \cdots, n-1$ and all $k = 1, \cdots, \bar{n}_4$. First ETE and Step 2 imply $\varphi_{iO_3}(P^{4,1}) = \frac{m_3}{n}$ for all $i = 1, \cdots, n-1$. Second we prove an induction: For any $k = 2, \cdots, \bar{n}_4$, if $\varphi_{iO_3}(P^{4,k-1}) = \frac{m_3}{n}$ for all $i = 1, \cdots, n-1$, then $\varphi_{iO_3}(P^{4,k}) = \frac{m_3}{n}$ for all $i = 1, \cdots, n-1$. Notice that $P^{4,k-1}$ and $P^{4,k}$ are different only in agent $(n-k+1)$'s preference. Then SP implies $\varphi_{n-k+1 O_3}(P^{4,k}) = \varphi_{n-k+1 O_3}(P^{4,k-1}) = \frac{m_3}{n}$. Hence feasibility and ETE imply $\varphi_{iO_3}(P^{4,k}) = \frac{m_3}{n}$ for all $i = 1, \cdots, n-1$.

Step 4: We show $\varphi_{iO_2}(P^{4,k}) = 0$ for all $i = n-k+1, \cdots, n-1$ and all $k = 2, \cdots, \bar{n}_4$. Fix a k and suppose without loss of generality $\varphi_{n-1 O_2}(P^{4,k}) = \beta > 0$. By ETE, $\varphi_{iO_2}(P^{4,k}) = \beta$ for all $i = n-k+1, \cdots, n-1$. Then Lemma 7 implies $\varphi_{iO_1}(P^{4,k}) = \frac{m_1+m_2}{n} - \beta$ for all $i = n-k+1, \cdots, n-1$ and EFF implies $\varphi_{iO_1}(P^{4,k}) = 0$ for all $i = 1, \cdots, n-k$. Then we have a contradiction against feasibility

$$
m_1 = (n-k) \times 0 + (k-1) \times \left(\frac{m_1+m_2}{n} - \beta\right) + \frac{m_1+m_2}{n} < k \times \frac{m_1+m_2}{n} \leqslant m_1
$$

where the last inequality comes from $k \leqslant \bar{n}_4 \leqslant \frac{m_1 n}{m_1+m_2}$.

Step 5: Lemma 7 and ETE imply all other probabilities we wanted. □

We now have a contradiction for the case where $\frac{m_2 n}{m_1+m_2}$ is an integer.
To see this, note that

$$\begin{aligned}P^{2,\bar{n}_2} =&(\hat{P}_1, \cdots, \hat{P}_{\frac{m_2 n}{m_1+m_2}-1}, P_{\frac{m_2 n}{m_1+m_2}}, \cdots, P_{n-1}, \bar{P}_n)\\ P^{4,\bar{n}_4} =&(\hat{P}_1, \cdots, \hat{P}_{n-\frac{m_1 n}{m_1+m_2}}, P_{n-\frac{m_1 n}{m_1+m_2}+1}, \cdots, P_{n-1}, \bar{P}_n)\\ =&(\hat{P}_1, \cdots, \hat{P}_{\frac{m_2 n}{m_1+m_2}}, P_{\frac{m_2 n}{m_1+m_2}+1}, \cdots, P_{n-1}, \bar{P}_n)\end{aligned}$$

Hence $P^{2,\bar{n}_2}$ and $P^{4,\bar{n}_4}$ are different only in agent $\frac{m_2 n}{m_1+m_2}$'s preference. Then SP implies $\varphi_{\frac{m_2 n}{m_1+m_2}O_3}(P^{2,\bar{n}_2}) = \varphi_{\frac{m_2 n}{m_1+m_2}O_3}(P^{4,\bar{n}_4})$, which implies a contradiction below.

$$\begin{aligned}&\varphi_{\frac{m_2 n}{m_1+m_2}O_3}(P^{2,\bar{n}_2}) = \varphi_{\frac{m_2 n}{m_1+m_2}O_3}(P^{4,\bar{n}_4})\\ \Leftrightarrow&\frac{m_3 - (\frac{m}{n} - \frac{m_1}{n-(\bar{n}_2-1)}) - (\bar{n}_2 - 1) \times \alpha(\bar{n}_2)}{n - \bar{n}_2} = \frac{m_3}{n}\\ \Leftrightarrow& - m_1 n(m_1 + m_2)((n + 1)m_1 + m_2) = 0\text{: contradiction!}\end{aligned}$$

To find the contradiction for the cases where $\frac{m_2 n}{m_1+m_2}$ is not an integer, we characterize the random allocations for the profiles in groups 5 and 6.

We present two types of random allocations and show that both of them are possible for the profiles in group 5. To begin with, let k^* be such that $k^* - 1 = n - \frac{m_1}{\frac{m}{n} - \frac{m_3}{2}}$, which is equivalent to $\frac{m}{n} - \frac{m_1}{n-(k^*-1)} - \frac{m_3}{2} = 0$.

Allocation 1:

	O_1	O_2	O_3
1	$-$	$-$	$\gamma(k)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k-1$	$-$	$-$	$\gamma(k)$
k	$-$	$-$	$\frac{m_3 - 2(\frac{m}{n} - \frac{m_1}{n-(k-1)}) - (k-1) \times \gamma(k)}{n-(k+1)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-2$	$-$	$-$	$\frac{m_3 - 2(\frac{m}{n} - \frac{m_1}{n-(k-1)}) - (k-1) \times \gamma(k)}{n-(k+1)}$
$n-1$	$\frac{m_1}{n-(k-1)}$	0	$\frac{m}{n} - \frac{m_1}{n-(k-1)}$
n	$\frac{m_1}{n-(k-1)}$	0	$\frac{m}{n} - \frac{m_1}{n-(k-1)}$

where $$\gamma(k) = \frac{\Gamma_1(k)m_1 + \Gamma_2(k)m_2 + \Gamma_3(k)m_3}{n[n^4 - 2(k+1)n^3 + (k^2 + 5k - 1)n^2 - (3k^2 + k - 2)n + 2(k^2 - k)]}$$

and $$\begin{aligned}\Gamma_1(k) &= 2(k-2)n^2 - 2(k^2 - k - 2)n + 2(k^2 - k - 2)\\ \Gamma_2(k) &= -2n^3 + 4kn^2 - 2(k^2 + k - 1)n + 2(k^2 - k)\\ \Gamma_3(k) &= n^4 - 2(k+1)n^3 + (k^2 + 5k - 1)n^2 - (3k^2 + k - 2)n + 2(k^2 - k)\end{aligned}$$

Allocation 2:

	O_1	O_2	O_3
1	$-$	$-$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$k-1$	$-$	$-$	0
k	$-$	$-$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-2$	$-$	$-$	0
$n-1$	$\frac{m_1}{n-(k-1)}$	$\frac{m}{n}-\frac{m_1}{n-(k-1)}-\frac{m_3}{2}$	$\frac{m_3}{2}$
n	$\frac{m_1}{n-(k-1)}$	$\frac{m}{n}-\frac{m_1}{n-(k-1)}-\frac{m_3}{2}$	$\frac{m_3}{2}$

Claim 5. *If $\frac{m_3}{m_2} \geqslant \frac{2}{n-2}$, $\varphi(P^{5,k})$ specifies probabilities on O_1, O_2, and O_3 as allocation 1 for all $k = 1, \cdots, \bar{n}_5$.*

Proof. Verification of the claim consists of four steps.

Step 1: We show, if $\frac{m_3}{m_2} \geqslant \frac{2}{n-2}$, $\varphi(P^{5,1})$ specifies probabilities on O_1, O_2, and O_3 as follows

	O_1	O_2	O_3
1	$\frac{m_1}{n}$	$\frac{m_2}{n-2}$	$\frac{m_2+m_3}{n}-\frac{m_2}{n-2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-2$	$\frac{m_1}{n}$	$\frac{m_2}{n-2}$	$\frac{m_2+m_3}{n}-\frac{m_2}{n-2}$
$n-1$	$\frac{m_1}{n}$	0	$\frac{m_2+m_3}{n}$
n	$\frac{m_1}{n}$	0	$\frac{m_2+m_3}{n}$

First, notice that $P^{5,1}$ and $P^{2,1}$ are different only in agent $(n-1)$'s preference. Then SP implies $\varphi_{n-1O_1}(P^{5,1}) = \varphi_{n-1O_1}(P^{2,1}) = \frac{m_1}{n}$ and hence feasibility and ETE imply $\varphi_{iO_1}(P^{5,1}) = \frac{m_1}{n}$ for all $i \in I$.

Second, we show $\varphi_{n-1O_2}(P^{5,1}) = \varphi_{nO_2}(P^{5,1}) = 0$. Suppose not, let $\beta \equiv \varphi_{n-1O_2}(P^{5,1}) = \varphi_{nO_2}(P^{5,1}) > 0$, then EFF implies $\varphi_{iO_3}(P^{5,1}) = 0$ for all $i = 1, \cdots, n-2$ and hence $\varphi_{n-1O_3}(P^{5,1}) = \varphi_{nO_3}(P^{5,1}) = \frac{m_3}{2}$. Then Lemma 7 requires $\frac{m_1+m_2+m_3}{n} = \frac{m_1}{n} + \beta + \frac{m_3}{2}$. Then $\beta > 0$ implies $\frac{m_1+m_2+m_3}{n} - \frac{m_1}{n} - \frac{m_3}{2} > 0$ which is equivalent to $\frac{m_3}{m_2} < \frac{2}{n-2}$: contradiction!

All the other probabilities we wanted are implied by Lemma 7 and ETE.

Step 2: We show $\varphi_{n-1O_3}(P^{2,k}) = \varphi_{nO_3}(P^{2,k}) = \frac{m}{n} - \frac{m_1}{n-(k-1)}$ for all $k = 1, \cdots, \bar{n}_5$.

Fix a k. First, notice that $P^{5,k}$ and $P^{2,k}$ are different only in agent $(n-1)$'s preference. Then SP implies $\varphi_{n-1O_1}(P^{5,k}) = \varphi_{n-1O_1}(P^{2,k}) = \frac{m_1}{n-(k-1)}$ and hence ETE implies $\varphi_{nO_1}(P^{5,k}) = \varphi_{n-1O_1}(P^{5,k}) = \frac{m_1}{n-(k-1)}$.

Second, we show $\varphi_{n-1O_2}(P^{5,k}) = \varphi_{nO_2}(P^{5,k}) = 0$. Suppose not, let $\varphi_{n-1O_2}(P^{5,k}) = \varphi_{nO_2}(P^{5,k}) = \beta > 0$, EFF implies $\varphi_{iO_3}(P^{5,k}) = 0$ for all $i = 1, \cdots, n-2$ and hence $\varphi_{n-1O_3}P^{5,k} = \varphi_{nO_3}P^{5,k} = \frac{m_3}{2}$. Then we have a contradiction:

$$\varphi_{nO_1}(P^{5,k}) + \varphi_{nO_2}(P^{5,k}) + \varphi_{nO_3}(P^{5,k}) = \varphi_{nO_1}(P^{5,1}) + \varphi_{nO_2}(P^{5,1}) + \varphi_{nO_3}(P^{5,1})$$

$$\Leftrightarrow \frac{m_1}{n-(k-1)} + \beta + \frac{m_3}{2} = \frac{m_1}{n} + 0 + \frac{m_2+m_3}{n}\text{: contradiction!}$$

where the contradiction comes from the fact that $\frac{m_1}{n-(k-1)} \geqslant \frac{m_1}{n}$, $\beta > 0$, and that $\frac{m_3}{m_2} \geqslant \frac{2}{n-2}$ implies $\frac{m_3}{2} \geqslant \frac{m_2+m_3}{n}$.

Last, Lemma 7 implies what we want.

Step 3: We show $\varphi_{1O_3}(P^{5,2}) = \gamma(2)$. Notice that $P^{5,2}$ and $P^{5,1}$ are different only in agent 1's preference. Then SP implies $\varphi_{1O_3}(P^{5,2}) = \varphi_{1O_3}(P^{5,1}) = \frac{m_2+m_3}{n} - \frac{m_2}{n-2} = \frac{-2m_2}{n(n-2)} + \frac{m_3}{n}$. Notice that $\Gamma_1(2) = 0$, $\Gamma_2(2) = -2n^3+8n^2-10n+4$, and $\Gamma_3(2) = n^4-6n^3+13n^2-12n+4$. Then

$$\begin{aligned}\gamma(2) &= \frac{0m_1 + (-2n^3+8n^2-10n+4)m_2 + (n^4-6n^3+13n^2-12n+4)m_3}{n[n^4-6n^3+13n^2-12n+4]}\\ &= \frac{-2m_2}{n(n-2)} + \frac{m_3}{n}.\end{aligned}$$

Step 4: We show an induction: For any $2 \leqslant k < \bar{n}_5$, if $\varphi_{iO_3}(P^{5,k}) = \gamma(k)$ for all $i = 1, \cdots, k-1$, then $\varphi_{iO_3}(P^{5,k+1}) = \gamma(k+1)$ for all $i = 1, \cdots, k$. By ETE, it suffices to show $\varphi_{kO_3}(P^{5,k+1}) = \gamma(k+1)$. Notice that $P^{5,k+1}$ and $P^{5,k}$ are different only in agent k's preference. Then

$$\begin{aligned}\varphi_{kO_3}(P^{5,k+1}) &= \varphi_{kO_3}(P^{5,k}) && \text{by SP}\\ &= \frac{m_3 - 2\times\varphi_{n-1O_3}(P^{5,k}) - (k-1)\times\varphi_{k-1O_3}(P^{5,k})}{n-(k+1)} && \text{by feasibility and ETE}\\ &= \frac{m_3 - 2(\frac{m}{n} - \frac{m_1}{n-(k-1)}) - (k-1)\times\gamma(k)}{n-(k+1)} && \text{by Step 2 and hypothesis assumption}\\ &= \gamma(k+1) && \text{by simplifying the expression}\end{aligned}$$

□

Claim 6. *If $\frac{m_3}{m_2} < \frac{2}{n-2}$ and $\bar{n}_5 < k^*$, $\varphi(P^{5,k})$ specifies probabilities on O_1, O_2, and O_3 as allocation 2 for each $k = 1, \cdots, \bar{n}_5$.*

Proof. Verification of the claim consists of four steps.

Step 1: We show, if $\frac{m_3}{m_2} < \frac{2}{n-2}$, $\varphi(P^{5,1})$ specifies probabilities on O_1, O_2, and O_3 as follows

	O_1	O_2	O_3
1	$\frac{m_1}{n}$	$\frac{m_2+m_3}{n}$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-2$	$\frac{m_1}{n}$	$\frac{m_2+m_3}{n}$	0
$n-1$	$\frac{m_1}{n}$	$\frac{m_2-(n-2)\times\frac{m_2+m_3}{n}}{2}$	$\frac{m_3}{2}$
n	$\frac{m_1}{n}$	$\frac{m_2-(n-2)\times\frac{m_2+m_3}{n}}{2}$	$\frac{m_3}{2}$

First by the same argument showing the Step 1 in Claim 5, $\varphi_{iO_1}(P^{5,1}) = \frac{m_1}{n}$.

Second we show $\varphi_{n-1O_2}(P^{5,1}) = \varphi_{n-1O_2}(P^{5,1}) > 0$. Suppose not, $\varphi(P^{5,1})$ is specified as by the Step 1 in Claim 5. Then $\varphi_{1O_3}(P^{5,1}) = \frac{m_2+m_3}{n} - \frac{m_2}{n-2} \geqslant 0$: contradicting against $\frac{m_3}{m_2} < \frac{2}{n-2}$.

Last, EFF implies $\varphi_{iO_3}(P^{5,1}) = 0$ for all $i = 1, \cdots, n-2$. All the other probabilities we wanted are implied by Lemma 7 and ETE.

Step 2: We show $\varphi_{n-1O_1}(P^{5,k}) = \varphi_{nO_1}(P^{5,k}) = \frac{m_1}{n-(k-1)}$ for all $k = 1, \cdots, \bar{n}_5$. Fix a k. Notice that $P^{5,k}$ and $P^{2,k}$ are different only in agent $(n-1)$'s preference. Then SP implies $\varphi_{n-1O_1}(P^{5,k}) = \varphi_{n-1O_1}(P^{2,k}) = \frac{m_1}{n-(k-1)}$ and hence ETE implies $\varphi_{nO_1}(P^{5,k}) = \varphi_{n-1O_1}(P^{5,k}) = \frac{m_1}{n-(k-1)}$.

Step 3: For any $k < k^*$, if $\varphi_{iO_3}(P^{5,k-1}) = 0$ for all $i = 1, \cdots, n-2$, then $\varphi_{iO_3}(P^{5,k}) = 0$ for all $i = 1, \cdots, n-2$. By EFF, it suffices to show $\varphi_{n-1O_2}(P^{5,k}) = \varphi_{nO_2}(P^{5,k}) > 0$. Suppose not. First, by Step 2 and Lemma 7, $\varphi_{n-1O_3}(P^{5,k}) = \varphi_{nO_3}(P^{5,k}) = \frac{m}{n} - \frac{m_1}{n-(k-1)}$. Second, notice that $P^{5,k}$ and $P^{5,k-1}$ are different only in agent k's preference. Then SP and ETE imply $\varphi_{iO_3}(P^{5,k}) = \varphi_{kO_3}(P^{5,k-1}) = 0$ for all $i = 1, \cdots, k$. Last, feasibility and ETE imply $\varphi_{iO_3}(P^{5,k}) = \frac{m_3 - 2\left(\frac{m}{n} - \frac{m_1}{n-(k-1)}\right)}{n-(k+1)}$. Then by $k < k^*$, we have a contradiction: $\varphi_{iO_3}(P^{5,k}) < \frac{m_3 - 2\left(\frac{m}{n} - \frac{m_1}{n-(k^*-1)}\right)}{n-(k+1)} = 0$.

□

Claim 7. *If $\frac{m_3}{m_2} < \frac{2}{n-2}$ and $\bar{n}_5 \geqslant k^*$, $\varphi(P^{5,k})$ specifies probabilities on O_1, O_2, and O_3 as allocation 2 for each $k = 1, \cdots, k^*$ and as allocation 1 for each $k = k^* + 1, \cdots, \bar{n}_5$.*

Proof. Verification of the claim consists of two steps.

By Claim 6, $\varphi(P^{5,k})$ specifies probabilities on O_1, O_2, and O_3 as allocation 2 for each $k = 1, \cdots, k^*$.

Step 1: $\varphi(P^{5,k^*})$ specifies probabilities on O_1, O_2, and O_3 as follows.

	O_1	O_2	O_3
1	$-$	$-$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
k^*-1	$-$	$-$	0
k^*	$-$	$-$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-2$	$-$	$-$	0
$n-1$	$\frac{m_1}{n-(k^*-1)}$	0	$\frac{m_3}{2}$
n	$\frac{m_1}{n-(k^*-1)}$	0	$\frac{m_3}{2}$

Step 2: For any $k > k^*$, if $\varphi_{n-1O_2}(P^{5,k-1}) = \varphi_{nO_2}(P^{5,k-1}) = 0$, then $\varphi_{n-1O_2}(P^{5,k}) = \varphi_{nO_2}(P^{5,k}) = 0$. Suppose not, then $\varphi_{iO_3}(P^{5,k}) = 0$ for all $i = 1, \cdots, n-2$ and hence $\varphi_{n-1O_3}(P^{5,k}) = \varphi_{nO_3}(P^{5,k}) = \frac{m_3}{2}$. By Step 2 and Lemma 7, $\varphi_{n-1O_2}(P^{5,k}) = \varphi_{nO_2}(P^{5,k}) = \frac{m}{n} - \frac{m_1}{n-(k-1)} - \frac{m_3}{2}$. Then by $k > k^*$, we have a contradiction: $\varphi_{n-1O_2}(P^{5,k}) = \varphi_{nO_2}(P^{5,k}) < \frac{m}{n} - \frac{m_1}{n-(k^*-1)} - \frac{m_3}{2} = 0$.

□

Claim 8. *For each preference profile $P^{6,k}$, $\varphi(P^{6,k})$ specifies probabilities on O_1, O_2, and O_3 as follows*

	O_1	O_2	O_3
1	$-$	$-$	$\frac{m_3}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-k$	$-$	$-$	$\frac{m_3}{n}$
$n-k+1$	$-$	$-$	$\frac{m_3}{n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-2$	$-$	$-$	$\frac{m_3}{n}$
$n-1$	$\frac{m_1+m_2}{n}$	0	$\frac{m_3}{n}$
n	$\frac{m_1+m_2}{n}$	0	$\frac{m_3}{n}$

Proof. Verification of the claim consists of three steps.

Step 1: We show $\varphi_{n-1O_1}(P^{6,k}) = \varphi_{nO_1}(P^{6,k}) = \frac{m_1+m_2}{n}$ for all $k = 2, \cdots, \bar{n}_6$. Fix a k. Notice that $P^{6,k}$ and $P^{4,k}$ are different only in agent $(n-1)$'s preference. Then SP implies $\varphi_{n-1O_1}(P^{6,k}) = \varphi_{n-1O_1}(P^{4,k}) = \frac{m_1+m_2}{n}$. Hence ETE implies $\varphi_{n-1O_1}(P^{6,k}) = \varphi_{nO_1}(P^{6,k}) = \frac{m_1+m_2}{n}$.

Step 2: We show $\varphi_{n-1O_2}(P^{6,k}) = \varphi_{nO_2}(P^{6,k}) = 0$ and $\varphi_{n-1O_3}(P^{6,k}) = \varphi_{nO_3}(P^{6,k}) = \frac{m_3}{n}$ for all $k = 2, \cdots, \bar{n}_6$. Fix a k. By Lemma 7, it suffices to show $\varphi_{n-1O_2}(P^{6,k}) = \varphi_{nO_2}(P^{6,k}) = 0$. Suppose not, then EFF implies $\varphi_{iO_3}(P^{6,k}) = 0$ for all $i = 1, \cdots, n-2$ and hence feasibility and ETE imply $\varphi_{n-1O_3}(P^{6,k}) = \varphi_{nO_3}(P^{6,k}) = \frac{m_3}{2}$. Then $\frac{m_1+m_2}{n} + 0 + \frac{m_3}{2} > \frac{m}{n}$: contradiction against Lemma 7.

Step 3: We show $\varphi_{iO_3}(P^{6,k}) = \frac{m_3}{n}$ for all $i = 1 \in I$ and all $k = 3, \cdots, \bar{n}_6$.

We first show $\varphi_{iO_3}(P^{6,3}) = \frac{m_3}{n}$ for all $i = 1 \in I$. Notice that, by Step 2 and ETE, $\varphi_{n-2O_3}(P^{6,2}) = \frac{m_3}{n}$. Notice also that $P^{6,3}$ and $P^{6,2}$ are different only in agent $(n-2)$'s preference. Then SP implies $\varphi_{n-2O_3}(P^{6,3}) = \varphi_{n-2O_3}(P^{6,2}) = \frac{m_3}{n}$. Then Step 2 and ETE imply what we want.

Now we show an induction: for any $3 \leqslant k < \bar{n}_6$, if $\varphi_{iO_3}(P^{6,k}) = \frac{m_3}{n}$ for all $i = 1 \in I$, then $\varphi_{iO_3}(P^{6,k+1}) = \frac{m_3}{n}$ for all $i = 1 \in I$. Notice that $P^{6,k+1}$ and $P^{6,k}$ are different only in agent $(n-k)$'s preference. Then SP implies $\varphi_{n-kO_3}(P^{6,k+1}) = \varphi_{n-kO_3}(P^{6,k}) = \frac{m_3}{n}$. Hence Step 2 and ETE imply what we want.

$\square$

Now we have the contradiction for the case where $\frac{m_2 n}{m_1+m_2}$ is not an integer. To see this, note that

$$P^{5,\bar{n}_5} = (\hat{P}_1, \cdots, \hat{P}_{\left[\frac{m_2 n}{m_1+m_2}\right]}, P_{\left[\frac{m_2 n}{m_1+m_2}\right]+1}, \cdots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$$

$$P^{6,\bar{n}_6} = (\hat{P}_1, \cdots, \hat{P}_{n-\left[\frac{m_1 n}{m_1+m_2}\right]}, P_{n-\left[\frac{m_1 n}{m_1+m_2}\right]+1}, \cdots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n)$$

Notice that $\left[\frac{m_2 n}{m_1+m_2}\right] = \left[n - \frac{m_1 n}{m_1+m_2}\right] = \left(n - \left[\frac{m_1 n}{m_1+m_2}\right]\right) - 1$. Then $P^{5,\bar{n}_5}$ and $P^{6,\bar{n}_6}$ are different only in agent $\left(\left[\frac{m_2 n}{m_1+m_2}\right] + 1\right)$'s preference. Hence SP implies $\varphi_{\left[\frac{m_2 n}{m_1+m_2}\right]+1O_3}(P^{5,\bar{n}_5}) = \varphi_{\left[\frac{m_2 n}{m_1+m_2}\right]+1O_3}(P^{6,\bar{n}_6})$.

If $\varphi(P^{5,\bar{n}_5})$ is in the form of Allocation 2, the contradiction is evident: $0 \neq \frac{m_3}{n}$.

If $\varphi(P^{5,\bar{n}_5})$ is in the form of Allocation 1, the contradiction is given by Condition 2 in Appendix C.

$$\frac{m_3 - 2(\frac{m}{n} - \frac{m_1}{n-(\bar{n}_5-1)}) - (\bar{n}_5 - 1) \times \gamma(\bar{n}_5)}{n - (\bar{n}_5 + 1)} \neq \frac{m_3}{n}.$$

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