Random Assignments of Bundles*

Shurojit Chatterji[†] Peng Liu[‡]

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Abstract

We propose a model studying the random assignments of bundles with no free disposal. The key difference between our model and the one where objects are allocated (see Bogomolnaia and Moulin (2001)) is one of feasibility. The implications of this difference are significant. Firstly, the characterization of sd-efficient random assignments is more complex. Secondly, we are able to identify a preference restriction, called essential monotonicity, under which the random serial dictatorship rule (extended to the setting with bundles) is equivalent to the probabilistic serial rule (extended to the setting with bundles). This equivalence implies the existence of a rule on this restricted domain satisfying sd-efficiency, sd-strategy-proofness, and equal treatment of equals. Moreover, this rule only selects random assignments which can be decomposed as convex combinations of deterministic assignments.

Keywords: Random assignments; bundles; decomposability; sd-efficiency; sd-strategy-proofness; equal treatment of equals

JEL Classification: C78, D47, D71.

1 Introduction

We study the problem of allocating a finite set of objects to a finite set of agents, where money transfers are prohibited and each agent receives a bundle of objects. Each object has a certain number of identical copies. We refer to this number as the capacity of the object and require that it be smaller than the number of agents. A bundle is a subset of objects that contains at most one copy of each object. In order to preserve fairness, we adopt randomization in allocations.[1]

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†School of Economics, Singapore Management University, Singapore.

‡Faculty of Economics and Management, East China Normal University, Shanghai 200062, China.

[1] Pápai (2000b) and Pápai (2001) consider allocations in bundles rather than in individual objects. However, these studies did not adopt randomization.

A central assumption in earlier studies on random assignments is that each agent gets at most one object.[2] However, for many relevant applications, it is more appropriate to allocate objects in bundles. One reason is that complementarity may require allocation in bundles in order to improve efficiency. Another reason is that the total number of objects may be more than the number of agents and free disposal may not be acceptable, as for instance is the case when a number of tasks exceeding the number of agents have to be accomplished. In this paper, we propose a model studying the random assignment of bundles in the absence of free disposal; by requiring that all copies of every object be allocated, we distinguish our model from that of other studies, for example, the course allocation problem, where seats of a course may be freely disposed of.[3] Another feature of our model is that we do not associate with objects priorities over agents, which distinguishes our model from the literature on school choice where priorities are essential (see Abdulkadiroğlu and Sönmez (2003)).

Under our formulation, a random assignment can be identified with a plane stochastic matrix which reflects the feature that feasibility in our model requires that groups of columns sum to fixed integers in a combinatorial fashion. The consequences of this more complex feasibility requirement are that not every random assignment is decomposable as a lottery over deterministic assignments, and that furthermore, the characterization of efficient random assignments for the setting of object allocation, (see Bogomolnaia and Moulin (2001)), is no longer valid.[4]

Our first result is a characterization of sd-efficiency using a condition called unbalancedness. We next extend the Probabilistic Serial rule (henceforth the PSB rule) and the Random Serial Dictatorship rule, (henceforth the RSDB rule) to the setting with bundles and observe that the RSDB rule satisfies sd-strategy-proofness, decomposability, and equal treatment of equals but violates sd-efficiency, while the PSB rule satisfies sd-efficiency and equal treatment of equals but violates sd-strategy-proofness and decomposability. As our principal finding, we identify a particular domain of preferences, the domain of essentially monotonic preferences, for which there exists a random assignment rule which selects only decomposable assignments and which satisfies additionally sd-strategy-proofness, sd-efficiency, and equal treatment of equals. Finally we demonstrate the non-existence of a sd-strategy-proof, sd-efficient, and sd-envy-free rule on the universal domain under a technical condition.[5]

In order to define the domain of essentially monotonic preferences, we first let agents se-

[2]Hylland and Zeckhauser (1979) were the first to introduce randomization to the allocations of indivisible objects. Abdulkadiroğlu and Sönmez (1998) proved the Random Serial Dictatorship rule to be equivalent to randomization over core allocations. Bogomolnaia and Moulin (2001) pointed out the ex ante inefficiency of the Random Serial Dictatorship rule, introduced simultaneous eating algorithms to identify all sd-efficient random assignments, showed that the Random Serial Dictatorship rule is sd-strategy-proof and that the Probabilistic Serial rule is sd-efficient but not sd-strategy-proof.

[3]The papers of Budish (2011), Sönmez and Ünver (2010), and Budish and Cantillon (2012) all assume free disposal, which is appropriate for their study of course allocations in universities, and focus on the so-called "pseudo-market" approach which endows the agents with pseudo money and then uses a mechanism that mimics the market equilibrium. Alon et al. (2015) considers a model where certain pairs of agents are couples and the two agents in a couple are required to receive the same lottery, a restriction not imposed in our model.

[4]Budish et al. (2013) studies the decomposability of a random assignment under an assumption called bihierarchy, which is violated in our set up.

[5]Similar impossibilities in the setting of object allocation can be found in Bogomolnaia and Moulin (2001), Kasajima (2013), Chang and Chun (2017), and Liu and Zeng (2019).

quentially take bundles which contain exactly one copy of each object that is still available. The set of bundles so dispensed are referred to as critical.[6] Essential monotonicity requires that a bundle contained in a critical bundle is less preferred to this critical bundle. In order to show the existence of a desirable random assignment rule, we show that the PSB and the RSDB rules are equivalent on the essentially monotonic domain and furthermore, degenerate to a constant rule. Given the disparity these two rules display in the setting with objects, their equivalence comes as a surprise as does the fact that sd-efficiency is preserved by a simple constant assignment on such a large domain, a finding not common in mechanism design.

The paper is organized as follows. Section 2 defines formally the model and axioms, the RSDB rule and the PSB rule, and introduces the domain of essentially monotonic preferences. Section 3 contains our results. Some discussions and examples that are omitted from the main exposition are gathered in Section 4. Section 5 contains all proofs except the proof of a Lemma which is of independent interest and appears in the appendix.

2 Preliminaries

2.1 Model and Axioms

Let $I \equiv \{1, \cdots, n\}$ denote a finite set of agents and let X denote a finite set of objects. Let in addition $m \equiv |X|$. Assume $n \geqslant 2$ and $m \geqslant 2$. In order to incorporate situations where some objects are physically identical, we allow an object to have multiple copies. For each $x \in X$, the capacity of x is a positive integer q_x, which denotes the number of its copies. We assume $q_x \in \{1, 2, \cdots, n-1\}$ so that the number of copies is smaller than the number of agents.[7] The capacities are collected in a vector $q = (q_x)_{x \in X}$.

A bundle of objects is a subset of X. The set of bundles is hence the power set 2^X and denoted as $\mathcal{X}$. Note that our definition of a bundle does not allow it to contain more than one copy of any object. This assumption is reasonable in the context of the applications of our model mentioned in the Section 4.1. Throughout the paper we denote objects with lowercase English alphabets and denote bundles with uppercase English alphabets, i.e., $a, b, c, x, y, z \in X$ and $A, B, C \in \mathcal{X}$. In addition, we usually denote the bundle $\{a, b, c\}$ simply as abc.

Each agent $i \in I$ is assumed to have a strict preference P_i on bundles, i.e., a linear order on $\mathcal{X}$. Following the convention, we denote $A\ R_i\ B$ if and only if either $A = B$ or $A\ P_i\ B$. The set of all strict preferences is denoted as $\mathbb{P}$ and referred to as the universal domain. Let $\mathbb{D} \subset \mathbb{P}$ be a nonempty subset of the universal domain. We treat this given subset as the set of admissible preferences and call it **the domain** of the problem. Given an arbitrary nonempty

[6]For example, if there are 10 copies each of a and b and 5 copies each of c and d, then each of the first 5 agents take the grand bundle $abcd$; each of the next 5 agents take ab; and all the others take the empty bundle. The critical bundles are accordingly $abcd$, ab, and the empty bundle.

[7]This assumption is reasonable for many relevant applications where the number of agents is large. The cases where $q_x \geqslant n$ for some x can be transformed into our setting as follows. Suppose there are 6 copies of a and only 5 agents, we can split these six copies into smaller groups and treat each group as one type of an object. For example, we may treat them as two objects, each of which has 3 copies. Depending on how one splits and allocates these copies, one obtains a different model. To each such model, our methodology applies.

subset of bundles $\bar{\mathcal{X}} \subset \mathcal{X}$ and an arbitrary preference $P_i \in \mathbb{P}$, denote $r_k(P_i, \bar{\mathcal{X}})$ as the k-th ranked bundle in $\bar{\mathcal{X}}$ according to P_i, i.e., $|\{A \in \bar{\mathcal{X}} : A \; R_i \; r_k(P_i, \bar{\mathcal{X}})| = k$.

A deterministic assignment can be presented as a matrix, whose rows are associated with agents and columns associated with bundles. The elements are either zeros or ones, where "one" means the corresponding agent gets the corresponding bundle and "zero" means she does not. Each agent gets exactly one bundle, which means every row of the matrix has exactly one non-zero element. Notice that this does not mean every agent will get some object, since the empty set is also treated as a bundle, i.e., $\emptyset \in \mathcal{X}$. In addition, an object $x \in X$ with capacity q_x is allocated to exactly q_x agents. We therefore impose **no free disposal** in our environment. Deterministic assignments are formally defined below.

Definition 1. *A **deterministic assignment** is a matrix $D \in \{0,1\}^{I \times \mathcal{X}}$ such that*

1. $\forall \, i \in I$: $\sum_{A \in \mathcal{X}} D_{iA} = 1$,
2. $\forall \, x \in X$: $\sum_{i \in I, x \in A} D_{iA} = q_x$.

The set of deterministic assignments is denoted $\mathcal{D}$. If one restricts attention to deterministic assignments, one would expect that, in general, the agents with the same preference will be treated unequally. To allow for greater flexibility in design to deal with the fairness issue, we allow the elements of an assignment to be fractional numbers between zero and one, as below.

Definition 2. *A **random assignment** is a matrix $L \in [0,1]^{I \times \mathcal{X}}$ such that*

1. $\forall \, i \in I$: $\sum_{A \in \mathcal{X}} L_{iA} = 1$,
2. $\forall \, x \in X$: $\sum_{i \in I, x \in A} L_{iA} = q_x$.

The set of random assignments is denoted $\mathcal{L}$. It is evident that $\mathcal{D} \subset \mathcal{L}$. The following is an example of a specific random assignment.

Example 1. *Let $I = \{1,2,3\}$, $X = \{a,b\}$, $q_a = 1$, and $q_b = 2$. Figure 1 below depicts a random assignment.*

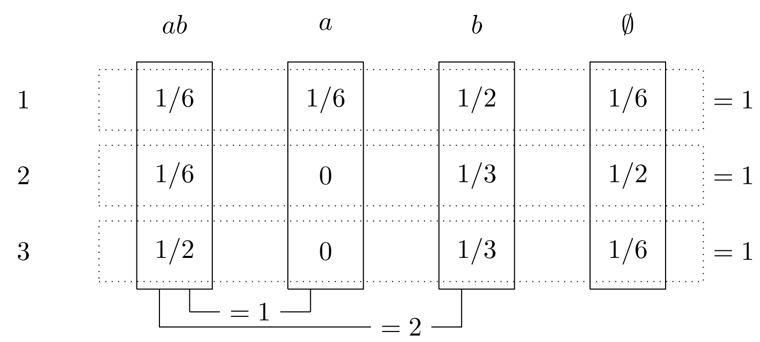


Figure 1: A Random Assignment

■

The fractional numbers in a random assignment are interpreted as the probability of the corresponding agent getting the corresponding bundle. Hence a row associated to agent i, denoted L_i, gives the lottery over bundles for agent i. In the random assignment specified above, L_2 specifies that agent 2 will get bundle ab with probability $1/6$, b with probability $1/3$, and empty bundle with probability $1/2$.

For deterministic assignments, condition 2 in the definition simply imposes ex post feasibility. For random assignments, the situation is more complicated. To fully interpret it, we need to introduce another notion below.

Definition 3. *A random assignment $L \in \mathcal{L}$ is **decomposable** if there is a lottery over deterministic assignments $\beta \in \Delta(\mathcal{D})$ such that*

$$L = \sum_{D \in \mathcal{D}} \beta(D) \cdot D,$$

where $\beta(D)$ denotes the probability lottery β assigns to D.

Such a lottery β is called a **decomposition** of L. Generally a decomposable random assignment may have multiple decompositions. The following is an example of a decomposition.

Example 2. *The random assignment L in Example 1 can be decomposed as follows*

$$\begin{pmatrix} & ab & a & b & \emptyset \\ 1: & 1/6 & 1/6 & 1/2 & 1/6 \\ 2: & 1/6 & 0 & 1/3 & 1/2 \\ 3: & 1/2 & 0 & 1/3 & 1/6 \end{pmatrix} = 1/2 \begin{pmatrix} & ab & a & b & \emptyset \\ 1: & 0 & 0 & 1 & 0 \\ 2: & 0 & 0 & 0 & 1 \\ 3: & 1 & 0 & 0 & 0 \end{pmatrix} + 1/6 \begin{pmatrix} & ab & a & b & \emptyset \\ 1: & 1 & 0 & 0 & 0 \\ 2: & 0 & 0 & 1 & 0 \\ 3: & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$+ 1/6 \begin{pmatrix} & ab & a & b & \emptyset \\ 1: & 0 & 0 & 0 & 1 \\ 2: & 1 & 0 & 0 & 0 \\ 3: & 0 & 0 & 1 & 0 \end{pmatrix} + 1/6 \begin{pmatrix} & ab & a & b & \emptyset \\ 1: & 0 & 1 & 0 & 0 \\ 2: & 0 & 0 & 1 & 0 \\ 3: & 0 & 0 & 1 & 0 \end{pmatrix} \quad \blacksquare$$

For a decomposable random assignment, Condition 2 in Definition 2 requires that, for x, the expected number of its copies that will be assigned to agents, through feasible deterministic assignments, is exactly q_x. In this sense, condition 2 imposes ex ante feasibility. We will on occasion call a random assignment feasible in order to emphasize the conditions in Definition 2.

In our setting with bundles, not every random assignment is decomposable (see Example 15). Since every lottery $\beta \in \Delta(\mathcal{D})$ specifies a random assignment $L \in \mathcal{L}$, the set $\mathcal{L}$ is strictly larger than the set of matrices specified by the lotteries in $\Delta(\mathcal{D})$.

A random assignment rule is formally defined as a mapping which selects a random assignment for every profile of admissible preferences.[8]

[8]As the set $\mathcal{L}$ is strictly larger than $\Delta(\mathcal{D})$, we allow a rule to select a non-decomposable random assignment. Indeed some important assignment rules like the probabilistic serial rule do so, and are included in our model. Although we do not impose decomposability on the definition of random assignment rules, we treat it as an axiom and impose it on our search for a desirable rule. In particular, our existence result (Theorem 1) identifies a rule that selects only decomposable random assignments. For the cases where an indecomposable random assignment is chosen, a decomposable approximation may be adopted (see Nguyen et al. (2016) and Akbarpour and Nikzad (2017)).

Definition 4. *A **random assignment rule** is a mapping* $\varphi : \mathbb{D}^n \to \mathcal{L}$.

The remainder of the section introduces four axioms that we impose on a desirable random assignment rule.

The first axiom concerns itself with decomposability, which is desirable since decomposable random assignments, being expressible as a lottery over deterministic assignments, are easier to operationalize. We call a random assignment rule decomposable if it selects only among decomposable random assignments. Formally, a random assignment rule $\varphi : \mathbb{D}^n \to \mathcal{L}$ is **decomposable** if $\varphi(P)$ is decomposable for every $P \in \mathbb{D}^n$.

In addition to decomposability, we impose three normative axioms on a desirable random assignment rule. The first deals with fairness and requires that whenever two agents report the same preference, they get the same lottery. Formally, a rule $\varphi : \mathbb{D}^n \to \mathcal{L}$ satisfies **equal treatment of equals** (or **ETE**) if for all $P \in \mathbb{D}^n$, $[P_i = P_j] \Rightarrow [\varphi_i(P) = \varphi_j(P)]$.

The second deals with efficiency and the third deals with incentive compatibility. However, both of these require an assumption on how an agent compares lotteries when she is identified by a preference on bundles. We thus need to extend a preference P_i over bundles $\mathcal{X}$ to a preference over lotteries in $\Delta(\mathcal{X})$. Following the standard approach, we adopt the stochastic dominance extension, which assumes that a lottery $L_i \in \Delta(\mathcal{X})$ is at least as good as $L'_i \in \Delta(\mathcal{X})$ if, for each bundle $A \in \mathcal{X}$, the probability of getting a bundle that is at least as good as A given by L_i is no less than that given by L'_i.[9] Formally,

Definition 5. *Given* $P_i \in \mathbb{P}$, $L_i \in \Delta(\mathcal{X})$ ***stochastically dominates*** $L'_i \in \Delta(\mathcal{X})$, *denoted as* $L_i \ P_i^{sd} \ L'_i$, *if for all* $B \in \mathcal{X}$

$$\sum_{AR_iB} L_{iA} \geqslant \sum_{AR_iB} L'_{iA}.$$

With the stochastic dominance extension, we define the remaining two axioms. An assignment L is sd-efficient at $P \in \mathbb{D}^n$ if there exists no $L' \in \mathcal{L}$ that Pareto dominates L, i.e., $L' \neq L$ and $L'_i \ P_i^{sd} \ L_i$ for all $i \in I$. Accordingly, a rule $\varphi : \mathbb{D}^n \to \mathcal{L}$ is **sd-efficient** if $\varphi(P)$ is sd-efficient at P, for all $P \in \mathbb{D}^n$. We address the general question of whether a random assignment is sd-efficient at a given profile in Section 3.1. Finally, a rule is sd-strategy-proof if truth-telling is always a weakly dominant strategy in the associated preference revelation game. Formally, a rule $\varphi : \mathbb{D}^n \to \mathcal{L}$ is **sd-strategy-proof** if for all $i \in I$, $P \in \mathbb{D}^n$, and $P'_i \in \mathbb{D}$, $\varphi_i(P_i, P_{-i}) \ P_i^{sd} \ \varphi_i(P'_i, P_{-i})$. We say a rule is **desirable** if it satisfies decomposability, sd-strategy-proofness, sd-efficiency, and equal treatment of equals.

2.2 The Random Serial Dictatorship Rule for Bundles

In the classical random assignment model, the random serial dictatorship rule (Abdulkadiroğlu and Sönmez (1998)) is defined as the equally weighted average of serial dictatorship rules (Svensson (1999)), each of which is a deterministic rule parameterized by an ordering of agents. Such an ordering is defined as a one-to-one mapping $\sigma : \{1, \cdots, n\} \to I$, where $\sigma(1)$

[9]This assumption is equivalent to assuming that a lottery L_i is at least as good as L'_i if and only if, for every Bernoulli utility representing P_i, L_i gives an expected utility that is at least as high as that is given by L'_i.

denotes the agent ordered the first, $\sigma(2)$ the second, and so on. The corresponding serial dictatorship rule lets the agents pick their respectively favorite objects sequentially. In particular, $\sigma(1)$ gets her favorite object, $\sigma(2)$ gets her favorite within the remaining objects, and so on.

For the setting with bundles, a seemingly natural extension of the serial dictatorship rule is one where every agent takes a bundle rather than an object. However, the following example indicates that such an extension may specify an infeasible assignment because of its conflict with the no free disposal requirement.

Example 3. *Consider the situation where $X = \{a, b\}$, $q_a = q_b = 1$, and $I = \{1, 2\}$. Consider a preference profile where agents have the same preference: $a \succ \emptyset \succ ab \succ b$. Let σ be an ordering of agents such that $\sigma(1) = 1$ and $\sigma(2) = 2$.*

In the first step, the set of available bundles is $\{ab, a, b, \emptyset\}$. Hence agent 1 will take a. Then the set of available bundles for agent 2 will be $\{b, \emptyset\}$, from which agent 2 will choose $\emptyset$. Then the deterministic assignment will be $D_{1a} = D_{2\emptyset} = 1$. However, this is not feasible since b is not assigned. ■

To deal with the problem, we introduce for each object $x \in X$, an "opposite object", denoted as $\bar{x}$, and refer to it as "not x." In addition, we will say $\bar{x} \in A$ if $x \notin A$. For each $\bar{x}$, we define its capacity as $n - q_x$ and whenever an agent takes a bundle A which does not contain x, we deduct the available units of $\bar{x}$ by one. Accordingly, we define a bundle A as **available**, if for every $x \in X$, $x \in A$ implies that there are still some units of x available and $x \notin A$ implies that there are still some units of $\bar{x}$ available. This rules out the infeasible assignments seen in Example 3 since whenever an agent takes a bundle not containing x, the available units of each opposite object $\bar{x}$ will be less. So eventually, when no more $\bar{x}$ is available, subsequent agents have to take x.

We present the serial dictatorship rule for bundles on an arbitrary domain below, where q_x^{v-1} and $q_{\bar{x}}^{v-1}$ denote respectively the available units of x and $\bar{x}$ for the v-th agent, which then defines as $\mathcal{X}^{v-1}$ the available bundles.

Definition 6. ***Serial dictatorship for bundles (SDB)*** *is a deterministic assignment rule $SDB^\sigma : \mathbb{D}^n \to \mathcal{D}$ parameterized by an ordering of agents $\sigma : \{1, 2, \cdots, n\} \to I$, such that given a preference profile $P \in \mathbb{D}^n$, $SDB^\sigma(P) = D$, specified by the following.*

Let $\mathcal{X}^0 = \mathcal{X}$, $q_x^0 = q_x$, and $q_{\bar{x}}^0 = n - q_x$, for all $x \in X$.

For $v = 1, \cdots, n$,

$$D_{\sigma(v)A} = \begin{cases} 1 & \text{if } A = r_1(P_{\sigma(v)}, \mathcal{X}^{v-1}) \\ 0 & \text{otherwise} \end{cases}$$

$$q_x^v = \begin{cases} q_x^{v-1} - 1 & \forall x \in r_1(P_{\sigma(v)}, \mathcal{X}^{v-1}) \\ q_x^{v-1} & \text{otherwise} \end{cases} \qquad q_{\bar{x}}^v = \begin{cases} q_{\bar{x}}^{v-1} - 1 & \forall x \notin r_1(P_{\sigma(v)}, \mathcal{X}^{v-1}) \\ q_{\bar{x}}^{v-1} & \text{otherwise} \end{cases}$$

$$\mathcal{X}^v = \mathcal{X}^{v-1} \backslash \{A \in \mathcal{X}^{v-1} : \exists x \in X \text{ s.t. } [x \in A, q_x^v = 0] \text{ or } [x \notin A, q_{\bar{x}}^v = 0]\}.$$

To illustrate that the SDB rule is well-defined, we present the following example.

Example 4. *Consider the setting of Example 3. The capacities of the objects are as follows:*

$$q^0_a = 1, \quad q^0_b = 1, \quad q^0_{\bar{a}} = 1, \quad q^0_{\bar{b}} = 1.$$

For agent 1*, the set of available bundles is* $\mathcal{X}^0 = \{ab, a, b, \emptyset\}$*, from which she takes* a*. Then the capacities of objects will be updated as follows:*

$$q^1_a = q^0_a - 1 = 0, \quad q^1_b = q^0_b = 1, \quad q^1_{\bar{a}} = q^0_{\bar{a}} = 1, \quad q^1_{\bar{b}} = q^0_{\bar{b}} - 1 = 0.$$

Hence the set of available bundles for agent 2 *is* $\mathcal{X}^1 = \{b\}$*. This indicates that agent* 2 *has to take* b *and the final assignment is such that agents* 1 *and* 2 *get respectively* a *and* b*.* ■

With the above well-defined serial dictatorship rules, we define the random serial dictatorship rule as the equally weighted combination of these deterministic rules. Let Σ denote the set of all orderings of agents.

Definition 7. ***Random serial dictatorship for bundles (RSDB)*** *is a random assignment rule* $RSDB : \mathbb{D}^n \to \mathcal{L}$ *such that given a preference profile* $P \in \mathbb{D}^n$*,*

$$RSDB(P) = \frac{1}{|\Sigma|} \sum_{\sigma \in \Sigma} SDB^{\sigma}(P).$$

The following Property, proved in Section 5.2, summarizes properties of the RSDB rule acting on the universal domain.

Property 1. *The RSDB rule on the universal domain satisfies decomposability, sd-strategy-proofness, equal treatment of equals but violates sd-efficiency.*

The performance of the RSDB rule on the universal domain and the underlying logic turn out to be the same as in the setting of object allocation (see Bogomolnaia and Moulin (2001)).

2.3 The Probabilistic Serial Rule for Bundles

The PS rule in the classical random assignment model (Bogomolnaia and Moulin (2001)) is a special case of the so-called simultaneous eating algorithm, and corresponds to the case where all agents eat at the uniform speed. It treats the objects as if they are infinitely divisible and proceeds as follows: all the agents eat their respectively favorite objects at the uniform speed, until some object is exhausted; thereafter, agents eat their respectively favorite objects among the available ones, still at the uniform speed, until some other object is exhausted; this procedure is repeated until all the objects are exhausted. Finally, the share of an object eaten by an agent is interpreted as the probability that this agent gets this object.

The PS rule can be naturally extended to the setting with bundles, where every agent eats a bundle rather than an object. In particular, an agent eating a bundle means that she simultaneously eats every object contained in that bundle. As in the definition of the SDB rule, here too we introduce for each $x \in X$ an opposite object $\bar{x}$. An agent eating a bundle A is equivalent to saying that she eats every x such that $x \in A$ and every $\bar{x}$ such that $x \notin A$. We define the probabilistic serial rule for bundles below, where r_x^{v-1} and $r_{\bar{x}}^{v-1}$ denote respectively the available shares of x and $\bar{x}$ for the v-th step. Accordingly, $\mathcal{X}^{v-1}$ denotes the available bundles for the v-th step. In particular, the length of the v-th step, i.e., $t^v - t^{v-1}$, is defined as the shortest time needed to exhaust at least one in x's and $\bar{x}$'s.

Definition 8. ***Probabilistic serial rule for bundles (PSB)*** *is a random assignment rule $PSB : \mathbb{D}^n \to \mathcal{L}$ such that given a preference profile $P \in \mathbb{D}^n$, $PSB(P) \equiv L^{\bar{v}}$ where $L^{\bar{v}}$ is generated by the following algorithm.*

Let $t^0 = 0$, $\mathcal{X}^0 = \mathcal{X}$, $r_x^0 = q_x$ and $r_{\bar{x}}^0 = n - q_x$ for all $x \in X$.

Let in addition L^0 be a matrix of size $n \times |\mathcal{X}|$ with all zeros.

For $v = 1, \cdots, \bar{v}$,

$$I_x^v \equiv \{i \in I : x \in r_1(P_i, \mathcal{X}^{v-1})\}, \forall x \in X;$$

$$I_{\bar{x}}^v \equiv I \backslash I_x^v, \forall x \in X;$$

$$t^v \equiv t^{v-1} + \min \left\{ \left\{ \frac{r_x^{v-1}}{|I_x^v|} : r_x^{v-1} > 0 \right\} \bigcup \left\{ \frac{r_{\bar{x}}^{v-1}}{|I_{\bar{x}}^v|} : r_{\bar{x}}^{v-1} > 0 \right\} \right\};$$

$$L_{iA}^v \equiv L_{iA}^{v-1} + \begin{cases} t^v - t^{v-1}, & \textit{if } A = r_1(P_i, \mathcal{X}^{v-1}) \\ 0, & \textit{otherwise} \end{cases}, \forall i \in I, A \in \mathcal{X}^{v-1};$$

$$r_x^v \equiv r_x^{v-1} - (t^v - t^{v-1}) \cdot |I_x^v|, \forall x \in X;$$

$$r_{\bar{x}}^v \equiv r_{\bar{x}}^{v-1} - (t^v - t^{v-1}) \cdot |I_{\bar{x}}^v|, \forall x \in X;$$

$$\mathcal{X}^v \equiv \mathcal{X}^{v-1} \backslash \{A \in \mathcal{X}^{v-1} : \exists x \in X \textit{ s.t. } [x \in A, r_x^v = 0] \textit{ or } [x \notin A, r_{\bar{x}}^v = 0]\};$$

where $\bar{v}$ is identified by $\mathcal{X}^{\bar{v}} = \emptyset$.

The following example illustrates an eating procedure.

Example 5. *Let $I = \{1, 2, 3\}$, $X = \{a, b, c\}$, and $q_x = 1\ \forall x \in X$. Let the preference profile P be as below.*

$$\begin{array}{lcccc} P_1: & ab & abc & \cdots & \cdots \\ P_2: & \emptyset & ab & c & \cdots \\ P_3: & \emptyset & ab & c & \cdots \end{array} \qquad \begin{array}{lcccc} & \emptyset & ab & abc & c \\ L_1: & 0 & 2/3 & 1/3 & 0 \\ L_2: & 2/3 & 0 & 0 & 1/3 \\ L_3: & 2/3 & 0 & 0 & 1/3 \end{array}$$

Initially the available shares are $r_a^0 = r_b^0 = r_c^0 = 1$ and $r_{\bar{a}}^0 = r_{\bar{b}}^0 = r_{\bar{c}}^0 = 2$ and hence every bundle is available.

In the first period, agent 1 eats bundle ab and agents 2 and 3 eat $\emptyset$. So the sets of agents who eat various available objects are as follows:

$$\begin{array}{lll} I_a^1 = \{1\} & I_b^1 = \{1\} & I_c^1 = \emptyset \\ I_{\bar{a}}^1 = \{2, 3\} & I_{\bar{b}}^1 = \{2, 3\} & I_{\bar{c}}^1 = \{1, 2, 3\}. \end{array}$$

The object $\bar{c}$ will be exhausted first since $r_{\bar{c}}^0 / |I_{\bar{c}}^1| = 2/3$ is the smallest among available objects. This also identifies the end of the first period, i.e., $t^1 = 2/3$. Hence in the first period *agent* 1 *eats* $2/3$ *of* ab *and agents* 2 *and* 3 *each eats* $2/3$ *of* $\emptyset$. *We now update the available shares of objects as below*

$$\begin{aligned} r_a^1 &= r_a^0 - 2/3 \cdot |I_a^1| = 1/3 & r_{\bar{a}}^1 &= r_{\bar{a}}^0 - 2/3 \cdot |I_{\bar{a}}^1| = 2/3 \\ r_b^1 &= r_b^0 - 2/3 \cdot |I_b^1| = 1/3 & r_{\bar{b}}^1 &= r_{\bar{b}}^0 - 2/3 \cdot |I_{\bar{b}}^1| = 2/3 \\ r_c^1 &= r_c^0 - 2/3 \cdot |I_c^1| = 1 & r_{\bar{c}}^1 &= r_{\bar{c}}^0 - 2/3 \cdot |I_{\bar{c}}^1| = 0. \end{aligned}$$

So, except for $\bar{c}$, *all the other objects are still available, which defines the set of available bundles as* $\mathcal{X}^1 = \{abc, ac, bc, c\}$. *In particular, the set* $\emptyset$ *is not available any more.*

In the second period, agent 1 *eats* abc, *which is her favorite in* $\mathcal{X}^1$ *and agents* 2 *and* 3 *eat* c, *which is their favorite in* $\mathcal{X}^1$. *The sets of agents who eat various available objects are as follows:*

$$\begin{aligned} &I_a^2 = \{1\} & &I_b^2 = \{1\} & &I_c^2 = \{1,2,3\} \\ &I_{\bar{a}}^2 = \{2,3\} & &I_{\bar{b}}^2 = \{2,3\} & &I_{\bar{c}}^2 = \emptyset. \end{aligned}$$

Then all the objects will be exhausted at the same time $t^2 = t^1 + 1/3 = 1$ *since* $1/3 = r_a^1/|I_a^2| = r_b^1/|I_b^2| = r_c^1/|I_c^2| = r_{\bar{a}}^1/|I_{\bar{a}}^2| = r_{\bar{b}}^1/|I_{\bar{b}}^2|$. *In this period agent* 1 *eats* $1/3$ *of* abc *and each of agents* 2 *and* 3 *eats* $1/3$ *of* c. *At the end of the second period, all the objects are exhausted, the algorithm terminates, and the resulting random assignment is* L *as presented above.* ■

As the above example indicates, the PSB rule is well-defined. To interpret it, imagine that each agent has m mouths, each mouth corresponding to a particular object, say x, and that throughout the eating procedure, a mouth is devoted to either eating x or its opposite object $\bar{x}$.

The following Property, proved in Section 5.3, summarizes properties of the PSB rule acting on the universal domain.

Property 2. *The PSB rule on the universal domain satisfies sd-efficiency and equal treatment of equals but violates decomposability and sd-strategy-proofness.*

Except for the fact that the violation of decomposability is new, the performance of the PSB rule turns out to be the same as in the setting of object allocation. The underlying logic however is different since the characterization of sd-efficiency in the current setting is different (see Section 3.1). Moreover, due to the change in the feasibility requirement, the manipulation pattern turns out to be more complicated. An implication is that an existence result in the setting of object allocation (see Liu (2019)) becomes invalid. This observation is presented as Example 14 in Section 4.2.

2.4 Essentially Monotonic Preferences

In order to introduce our preference restriction, we identify first the following sequence of bundles and integers. We call them **critical bundles** and **critical capacities**.

$$
\begin{aligned}
&A_1 \equiv X, && d_1 \equiv \min\{q_x : x \in X\},\\
&A_2 \equiv \{x \in X : q_x > d_1\}, && d_2 \equiv \min\{q_x - d_1 : q_x > d_1\},\\
&\vdots && \vdots\\
&A_k \equiv \left\{x \in X : q_x > \textstyle\sum_{l=1}^{k-1} d_l\right\}, && d_k \equiv \min\left\{q_x - \textstyle\sum_{l=1}^{k-1} d_l : q_x > \textstyle\sum_{l=1}^{k-1} d_l\right\},\\
&\vdots && \vdots\\
&A_{K-1} \equiv \left\{x \in X : q_x > \textstyle\sum_{l=1}^{K-2} d_l\right\}, && d_{K-1} \equiv \min\left\{q_x - \textstyle\sum_{l=1}^{K-2} d_l : q_x > \textstyle\sum_{l=1}^{K-2} d_l\right\},\\
&A_K \equiv \left\{x \in X : q_x > \textstyle\sum_{l=1}^{K-1} d_l\right\}, && d_K \equiv n - \textstyle\sum_{l=1}^{K-1} d_l,
\end{aligned}
$$

where K is identified by $A_K = \emptyset$. It is evident that $d_{K-1} = \max\{q_x : x \in X\}$. By the structure above, $X = A_1 \supsetneqq A_2 \supsetneqq \cdots \supsetneqq A_{K-1} \supsetneqq A_K = \emptyset$.

Example 6. *Consider situations where objects have the same number of copies, i.e., $q_x = q_y$ for all $x, y \in X$. Then the critical bundles and capacities are as follows.*

$$
\begin{array}{ll}
A_1 = X & d_1 = q_x \\
A_2 = \emptyset & d_2 = n - q_x
\end{array}
$$

■

Example 7. *Consider a situation where $n = 6$, $X = \{a, b, c\}$, $q_a = 4$, $q_b = 3$, and $q_c = 2$. Then the critical bundles and capacities are as follows.*

$$
\begin{array}{ll}
A_1 = abc & d_1 = 2 \\
A_2 = ab & d_2 = 1 \\
A_3 = a & d_3 = 1 \\
A_4 = \emptyset & d_4 = 2.
\end{array}
$$

■

A preference is called essentially monotonic if whenever a bundle is a proper subset of a critical bundle, it is less preferred to this critical bundle. Formally

Definition 9. *A preference $P_i \in \mathbb{P}$ is **essentially monotonic** if for any critical bundle A_k and any $A \in \mathcal{X}$ such that $A \subsetneqq A_k$, $A_k \; P_i \; A$.*

Let $\mathbb{D}_{EM} \subset \mathbb{P}$ be the set of *all* essentially monotonic preferences and call it **the essentially monotonic domain**. As shown by Examples 6 and 7, the more the capacities vary, the greater the number of critical bundles. Hence more restrictions will be imposed on essentially monotonic preferences and $\mathbb{D}_{EM}$ will be smaller.

Among the preference restrictions studied in the setting with bundles, two are closely related to essentially monotonicity: **monotonicity** (Pápai, 2000b) and **separability** (Le Breton and Sen, 1999). A preference is monotonic if whenever a bundle is a proper subset of another bundle, the former is less preferred than the later. Formally, $\forall A, B \in \mathcal{X}$, $B \subsetneqq A \Rightarrow A \; P_i \; B$. A preference is separable if adding an additional object to a bundle is preferred if and only if the object itself

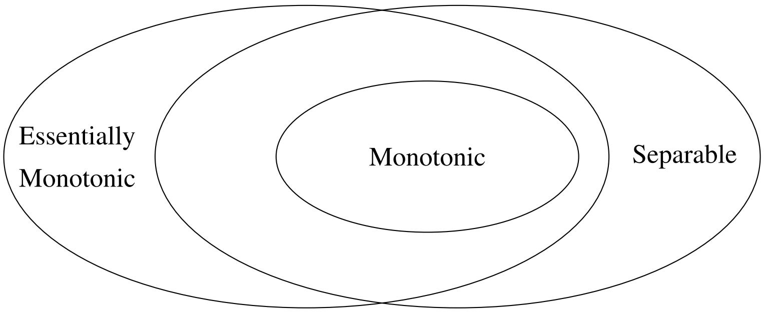


Figure 2: The Relationship among Preference Restrictions

is preferred to the empty bundle. Formally, $\forall A \in \mathcal{X}$ and $x \in X \backslash A$, $A \cup \{x\}\ P_i\ A$ if and only if $x\ P_i\ \emptyset$.[10]

Figure 2 shows the relationship. By definition, essential monotonicity is strictly weaker than monotonicity because the requirement that a bundle contained in another is less preferred is imposed only for bundles contained in critical bundles. This relation is true independent of the capacities. (Recall that the size of the essentially monotonic domain varies with the capacities.) Consider the critical bundles in Example 6, essential monotonicity requires only that the grand bundle, X, is the top ranked bundle. But monotonicity requires much more and hence the monotonic domain contains fewer preferences than the essentially monotonic domain. For another instance, consider the critical bundles in Example 7, essential monotonicity requires (i) the grand bundle X is top ranked, (ii) bundles a, b, and $\emptyset$ are less preferred than ab, and (iii) $\emptyset$ is less preferred than a. Hence essential monotonicity imposes less structure on preferences than does monotonicity.

The essentially monotonic domain and the separable domain overlap with each other but no one contains the other. This relation is true independent of the capacities. Consider Example 7 where there are totally 3 objects and 4 critical bundles. Recall that given the object set, these capacities identify a maximal set of critical bundles. In other words, the resulting essentially monotonic domain is minimal. Even in this case, there exists a preference, P_i below (where the critical bundles are underlined), which is essentially monotonic but not separable. To see that P_i is non-separable, notice that $\emptyset\ P_i\ c$ but $ac\ P_i\ a$. In addition P_i' below is a preference which is separable but not essentially monotonic.

$$P_i: \quad \underline{abc} \succ \underline{ab} \succ ac \succ bc \succ \underline{a} \succ b \succ \underline{\emptyset} \succ c$$

$$P_i': \quad bc \succ \underline{abc} \succ b \succ \underline{ab} \succ c \succ ac \succ \underline{\emptyset} \succ \underline{a}$$

3 Results

We first present a characterization of efficiency in the setting with bundles. We next present our main result which is an existence result for essentially monotonic domains. Finally, we

[10]The literature on matching with contracts employs a notion called "observable substitutability", which is a condition on choice functions introduced by Hatfield et al. (2017). However, essential monotonicity neither implies nor is implied by observable substitutability. Details are available on request.

present an impossibility result on the universal domain.

3.1 Efficiency

We address in this section the question of whether a given random assignment is sd-efficient at a given preference profile. For the problem of random assignments of objects, Bogomolnaia and Moulin (2001) provided two characterizations of sd-efficient assignments. The first says that a random assignment is sd-efficient at a profile if and only if a particular relation on objects is acyclic. The second is more mechanical and shows that at a profile, all sd-efficient random assignments can be found by the simultaneous eating algorithm with varying eating speeds. For the assignment problem of bundles, we find that neither characterization is true. In particular, acyclicity, while still necessary, is not sufficient.[11] A new condition called unbalancedness is provided and proved equivalent to sd-efficiency in the current setting.

We begin with a modified definition of acyclicity (Bogomolnaia and Moulin (2001)).

Definition 10. *A random assignment $L \in \mathcal{L}$ is **acyclic** at $P \in \mathbb{P}^n$ if and only if the relation $\tau(P, L)$ on $\mathcal{X}$ is acyclic where $A\ \tau(P, L)\ B \Leftrightarrow \exists i \in I$ such that $B\ P_i\ A$ and $L_{iA} > 0$.*

The next example shows that acyclicity is no longer sufficient to guarantee sd-efficiency.

Example 8. *Let $A = \{a, b, c\}$, $q = (1, 1, 1)$, $I = \{1, 2\}$. Let the preferences of two agents be*

$$\begin{array}{lcccccccc} P_1: & c & a & ab & b & \emptyset & bc & ac & abc \\ P_2: & a & c & ab & b & \emptyset & bc & ac & abc \end{array}$$

Consider random assignments L and L' below.

	c	a	ab	b	$\emptyset$	bc	ac	abc
L_1:	0	0	0.2	0	0.3	0	0	0.5
L_2:	0.2	0	0	0	0.5	0	0	0.3

	c	a	ab	b	$\emptyset$	bc	ac	abc
L'_1:	0.2	0	0	0.2	0.1	0	0	0.5
L'_2:	0	0.2	0	0	0.5	0	0	0.3

We claim that L above is acyclic at P. To see this, notice that if the relation $\tau(P, L)$ has a cycle, it must involve a preference reversal across two agents' preferences. According to P, agents' preferences coincide except between a and c. Hence a cycle of $\tau(P, L)$ requires at the same time $a\ \tau(P, L)\ c$ and $c\ \tau(P, L)\ a$. We show however $a\ \tau(P, L)\ c$ is not true. To see this, notice that agent 1 who prefers c to a has no positive probability of a and that agent 2 does not prefer c to a. Hence the relation $\tau(P, L)$ is acyclic. However L is not sd-efficient at P since it is Pareto dominated by L': $L' \neq L$, $L'_1\ P_1^{sd}\ L_1$, and $L'_2\ P_2^{sd}\ L_2$. ■

The failure of the acyclicity to suffice for sd-efficiency occurs as we are able to implement a sequence of probability transfers starting at L and leading to a feasible Pareto improvement. In particular, by comparing L and L' in the above example, we identify three probability transfers which construct L' from L, and illustrate them in Figure 3 below.

[11] As regards the simultaneous eating algorithm in the bundle setting, the working paper version (Chatterji and Liu (2018)) of this paper contains an example (Remark 4 on page 39), which shows that not all sd-efficient random assignments can be described using the class of simultaneous eating algorithms.

	c	a	ab	b	$\emptyset$	bc	ac	abc
L_1:	0	0	0.2	0	0.3	0	0	0.5
	$\alpha(1, ab, c) = 0.2$			$\alpha(1, ab, c) = 0.2$				
L_2:	0.2	0	0	0	0.5	0	0	0.3
	$\alpha(2, c, a) = 0.2$							

Figure 3: Probability Transfer System α

In particular, let $\alpha(1, ab, c) = 0.2$ denote the probability transfer of 0.2 from $(1, ab)$ to $(1, c)$ and let $\bar{L}$ denote the matrix resulting from such a probability transfer on L. Since c is preferred to ab by agent 1, she would prefer such a transfer. However, $\bar{L}$ is not a feasible random assignment. Specifically, $\sum_{i \in I, a \in A} \bar{L}_{iA} = q_a - 0.2$, $\sum_{i \in I, b \in A} \bar{L}_{iA} = q_b - 0.2$, and $\sum_{i \in I, c \in A} \bar{L}_{iA} = q_c + 0.2$. The net influence on the feasibility is as shown by the second column in the following table.

	$\alpha(1, ab, c) = 0.2$	$\alpha(1, \emptyset, b) = 0.2$	$\alpha(2, c, a) = 0.2$	Total
a	-0.2	0	$+0.2$	0
b	-0.2	$+0.2$	0	0
c	$+0.2$	0	-0.2	0

Next, we denote the remaining two transfers as respectively $\alpha(1, \emptyset, b) = 0.2$ and $\alpha(2, c, a) = 0.2$ and implement them successively starting from $\bar{L}$ to obtain L'. The third and fourth columns in the above table summarize the influence of these two transfers on feasibility. We see that the influence of the aforementioned transfers cancel out on each row, making L' a feasible random assignment.

To formalize the observation above, let $\mathcal{T} = I \times \mathcal{X} \times \mathcal{X}$. Then a **system of probability transfers** can be represented by a mapping $\alpha : \mathcal{T} \to \mathbb{R}_+$ that specifies for each triple (i, A, B) a non-negative number $\alpha(i, A, B)$, which denotes a probability transfer from (i, A) to (i, B). We require systems of probability transfers to be non-trivial in the sense that not all triples are assigned zero probability transfers. Given a random assignment $L \in \mathcal{L}$, a transfer system will construct a new matrix, denoted L', of size $|I| \times |\mathcal{X}|$. Formally $\forall j \in I$ and $C \in \mathcal{X}$,

$$L'_{jC} = L_{jC} + \sum_{\{(i,A,B) \in \mathcal{T} : i = j, B = C\}} \alpha(i, A, B) - \sum_{\{(i,A,B) \in \mathcal{T} : i = j, A = C\}} \alpha(i, A, B).$$

Generally such a matrix L' is not a feasible random assignment. In the definition below, we focus on a particular class of systems which not only construct feasible random assignments but also ensure that the assignments constructed dominate the original L at P.

Definition 11. *An assignment $L \in \mathcal{L}$ is **unbalanced** at $P \in \mathbb{P}^n$ if there is no $\alpha : \mathcal{T} \to \mathbb{R}_+$ s.t.*

(i) $\forall\ (i, A, B) \in \mathcal{T}$*:* $\alpha(i, A, B) > 0$ *implies* $L_{iA} > 0$ *and* $B\ P_i\ A$*,*

(ii) $\forall x \in X : \sum_{\{(i,A,B) \in \mathcal{T} : x \in B\}} \alpha(i, A, B) = \sum_{\{(i,A,B) \in \mathcal{T} : x \in A\}} \alpha(i, A, B)$.

*We say L is **balanced** at P if it is not unbalanced at P.*

The following proposition states that unbalancedness characterizes sd-efficiency.

Proposition 1. *Given $P \in \mathbb{P}^n$ and $L \in \mathcal{L}$, L is sd-efficient at P iff L is unbalanced at P.*

The formal proof is provided in Section 5.1.

Remark 1. If a random assignment L is dominated by another assignment L' at a particular preference profile P, we can classify this situation into one of the following two cases. Case 1: for each bundle, the corresponding columns in L and L' sum to the same number. Case 2: they sum to different numbers. It is easy to see that, for case 1, L is dominated if and only if it has a cycle defined by Bogomolnaia and Moulin (2001). However, for case 2, there could be various types of improvements. Besides the improvements illustrated in Figure 3, the following illustrates another type.

Let $I = \{1, 2\}$, $X = \{a, b, c\}$, and $q_a = q_b = q_c = 1$. Let P and L be as follows.

$$\begin{array}{lcccccccc} P_1: & abc & \emptyset & c & a & ab & b & bc & ac \\ L_1: & 0.5 & 0.3 & 0 & 0 & 0.2 & 0 & 0 & 0 \end{array}$$

$$\begin{array}{lcccccccc} P_2: & abc & \emptyset & a & c & ab & b & bc & ac \\ L_2: & 0.3 & 0.5 & 0 & 0.2 & 0 & 0 & 0 & 0 \end{array}$$

An improvement can be implemented as follows. First, transfer 0.1 probability from ab to c for agent 1 and 0.1 from c to ab for agent 2. Second, for agent 1, combine the probabilities of c and ab to 0.1 probability of abc and increase the probability of $\emptyset$ by 0.1. Last, do the same for agent 2. Then, the resulting random assignment is L' below, which dominates L at P.

$$\begin{array}{lcccccccc} P_1: & abc & \emptyset & c & a & ab & b & bc & ac \\ L'_1: & 0.6 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

$$\begin{array}{lcccccccc} P_2: & abc & \emptyset & a & c & ab & b & bc & ac \\ L'_2: & 0.4 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

The interesting feature of this improvement is that, in the first step, the probability transfer hurts agent 2 since 0.1 probability is transferred from a more preferred bundle (c) to a less preferred bundle (ab). However, this probability transfer allows for other probability transfers, which lead to a preferable final outcome. Our characterization, Proposition 1, simplifies the verification of sd-efficiency precisely because we dispense with the need to check all the various types of improvements. It suffices to focus only on probability transfers from less preferred bundles to more preferred bundles where the less preferred bundles have positive probabilities. ■

3.2 Existence

In the classical random assignment problem that allocates objects, two extensively studied rules are the random serial dictatorship rule (or RSD, see Abdulkadiroğlu and Sönmez (1998)) and the probabilistic serial rule (or PS, see Bogomolnaia and Moulin (2001)). It is well known that in the classical random assignment model, both rules treat equals equally. The PS rule is sd-efficient but not sd-strategy-proof while the RSD rule is sd-strategy-proof but not sd-efficient. Due to their distinct properties, these two rules are in general treated as competing alternatives for applications. We prove that these rules, adapted to the setting with bundles (as done in Sections 2.2 and 2.3 respectively), are equivalent on the domain of essentially monotonic preferences. This equivalence yields an existence result for desirable rules: there exists a decomposable, sd-efficient, sd-strategy-proof, and equal-treatment-of-equals random assignment rule on the essentially monotonic domain.

The equivalence obtains from the following critical feature of any profile of preferences where each agent's preference is drawn from the essentially monotonic domain $\mathbb{D}_{EM}$: Any pair $P_i, P_j \in \mathbb{D}_{EM}$ have the features that (i) the grand bundle comprising all objects is top ranked in each preference, and (ii) that the two preferences necessarily agree on the rankings of the critical bundles relative to each other.

For the PSB rule, essential monotonicity guarantees that at any point of time in the eating procedure, each agent's top-ranked bundle *within the available ones* is the same critical bundle. They consequently agree on their eating procedure and end up with the same lottery independently of their preferences. For the RSDB rule, essentially monotonicity guarantees the same lottery as the PSB rule as a consequence of the fact that the critical bundles are disbursed in the same order independent of the specific permutation used to order agents. We next illustrate the mechanics of the equivalence using an example.

Example 9. *Consider the setting in Example 7 where $n = 6$, $X = \{a, b, c\}$, $q_a = 4$, $q_b = 3$, and $q_c = 2$. Recall that the four critical bundles in this setting are abc, ab, a, and $\emptyset$. Suppose that the first three agents have identical preferences given by $\hat{P}_i$ while the last three agents have identical preferences given be $\tilde{P}_j$*

$$\begin{array}{lcccccccc} \hat{P}_i: & \underline{abc} & \underline{ab} & \underline{a} & \underline{\emptyset} & ac & bc & b & c \\ \tilde{P}_j: & \underline{abc} & ac & bc & c & \underline{ab} & b & \underline{a} & \underline{\emptyset} \end{array}$$

These two preferences may be seen as two extremes under the requirement of essential monotonicity. In particular, the first three agents treat all critical bundles better than the other bundles, while the others rank the critical bundles as low as possible without violating essential monotonicity. In particular, the relative ranking of the critical bundles remains the same across the two preferences.

Although the preferences of the two groups of agents are different, once the PSB rule is employed, their respective eating procedures turn out to be the same. At time 0, all agents start by eating abc. The first period ends (once c is depleted) at $1/3$. After $1/3$, agents 1, 2, and 3 eat ab, as this is their second-ranked bundle and it is available. For the remaining agents, their second, third and fourth ranked alternatives are unavailable as c has been depleted; they too therefore eat ab. Hence, in the second period, all agents eat ab and this period ends at $1/2$ when b is depleted. One can verify that all agents eat a in the third period, which ends at $2/3$, and $\emptyset$

in the fourth period, which ends at 1*. Consequently, all agents receive the same lottery below.*

$$\begin{array}{ccccccccc} & \underline{abc} & \underline{ab} & \underline{a} & \underline{\emptyset} & ac & bc & b & c \\ L_i: & 1/3 & 1/6 & 1/6 & 1/3 & 0 & 0 & 0 & 0 \end{array}$$

Next we turn to the RSDB rule. Given an arbitrary ordering of the set of agents, the first two agents under the ordering take the bundle abc*. Next, the third agent takes* ab *and the fourth takes* a*. Finally, the two remaining agents get* $\emptyset$*. Randomizing with uniform probability over all orderings over agents yields the lottery* L_i *above.* ■

We now state the following property, whose formal proof is provided in Section 5.4.

Property 3. *The RSDB rule is equivalent to the PSB rule on the essentially monotonic domain.*

The equivalence in Property 3 along with Property 1 and Property 2 gives the following result on the existence of a desirable rule.

Theorem 1. *There is a decomposable random assignment rule on the essentially monotonic domain satisfying sd-efficiency, sd-strategy-proofness, and equal treatment of equals.*

Remark 2. The proof of Property 3 generalizes the conclusion of Example 9 and shows that the PSB rule and the RSDB rule degenerate to the same constant rule. Therefore, given a specific problem, we simply identify the critical bundles and then allocate equally these bundles according the lottery specified in the proof of Property 3, regardless of the preferences of agents. As mentioned in the introduction, the fact that sd-efficiency is preserved by such a simple constant assignment on a large domain comes as a surprise. ■

3.3 Impossibility

This subsection presents an impossibility result on the universal domain. Recall that the capacities of objects are collected in a vector $q = (q_x)_{x \in X}$. Given q, we identify the critical bundles $A_1, \cdots, A_K$ and in particular denote K as the number of critical bundles. Since the grand bundle X and the empty bundle $\emptyset$ are identified as critical bundles no matter what the capacity vector q is, $K \geqslant 2$. In principle, the more capacities vary, the larger K is. Below, we present a general impossibility, which states that when there are at least four critical bundles, no rule on the universal domain satisfies sd-strategy-proofness, sd-efficiency, and sd-envy-freeness at the same time. Sd-envy-freeness is a fairness axiom stronger than equal treatment of equals and requires that an agent always weakly prefers her own lottery to any other's. Formally, a rule $\varphi : \mathbb{D}^n \to \mathcal{L}$ is **sd-envy-free** if $\forall P \in \mathbb{D}^n$ and $i, j \in I$, $\varphi_i(P)\ P_i^{sd}\ \varphi_j(P)$.

Proposition 2. *Given* $K \geqslant 4$*. There is no sd-strategy-proof, sd-efficient, and sd-envy-free rule on the universal domain.*

The proof is contained in Section 5.5.

Remark 3. Note that the impossibility applies when there are at least four critical bundles, i.e., $K \geqslant 4$. For the cases where $K = 2$ or 3, the picture is unclear and deserves further investigation. For example, consider $K = 2$, that is, all the objects have the same capacity $\bar{q}$. Then whether or not there is a desirable rule on the universal domain depends on the parameters of the problem m, n, and $\bar{q}$. When $m = n \geqslant 4$ and $\bar{q} = 1$, the impossibility of Bogomolnaia and Moulin (2001) (their Theorem 2 in particular) implies an impossibility in our setting. To see this, consider an arbitrary profile of preferences, where all singleton bundles are ranked at the top, and are followed by the empty bundle and then the other bundles. Then sd-efficiency implies that only singleton bundles get positive probabilities. Hence the proof of Theorem 2 in Bogomolnaia and Moulin (2001) can be used to prove the corresponding impossibility in our setting. However, for other cases, their proof can not be used. Take for example the case where $m = 5$, $n = 4$, and $\bar{q} = 1$. In this case, for a preference profile mentioned above, feasibility requires that some bundle containing more than one object gets positive probability. ■

4 Discussion and Examples

4.1 Two Applications of the Model

We present in this subsection two examples that illustrate the scope of our study.

Example 10. *A local government has developed a public housing project, which provides 50 units of standardized apartments, 50 parking slots, and 30 bicycle slots. These objects need to be fully allocated among 100 eligible applicants without any money transfers.* ■

The objects in the example above are publicly financed and consequently free disposal is not an option as the objects ought not to be wasted.[12] One way to allocate these objects is to do so via independent committees, one for each type of the object. However, given that the agents' preferences may reflect complementarity and substitution, the efficiency losses incurred by this method may be significant. It is therefore of interest to investigate the allocation of objects in bundles. Moreover, due to the indivisibility of the objects, agents reporting the same preference will potentially be treated differently. In order to achieve fairness, it is natural to resort to random assignments. In particular, every agent receives a lottery on bundles.

There are in total 8 distinct bundles. We denote an apartment as a, a parking slot as p, a bicycle slot as b and as before, a bundle is denoted simply as a sequence of these alphabets, for example, the grand bundle is denoted as apb rather than $\{a, p, b\}$.

Since objects are given to the agents for free, it might be reasonable to assume that a bundle containing more objects is better. This is captured by a classical preference restriction, monotonicity, which requires that a bundle contained in another bundle is less preferred to this later bundle. A typical monotonic preference is as follows.

$$P_i: \quad apb \succ ap \succ ab \succ pb \succ a \succ p \succ b \succ \emptyset$$

In order to identify our preference restriction of essential monotonicity, we proceed to identify the critical bundles for this example. Assume that agents are lined up and are required to

[12] Hence a treatment that replicates the rules used in the object assignment literature (see for example Pápai (2000a) and Bogomolnaia and Moulin (2001)) are not appropriate since a central assumption there is that each agent receives at most one object, which implies for our problem that finally 30 objects need to be discarded.

take bundles one by one. Moreover, whenever an agent is called upon, she takes one copy of each available object. As for the current case, each of the first 30 agents will take the grand bundle apb, each of the next 20 agents will take ap, and each of the remaining agents will take the empty bundle $\emptyset$. Hence, there are in total three critical bundles. Essential monotonicity imposes three preference restrictions: (i) a bundle contained in apb is less preferred; (ii) a bundle contained in ap is less preferred; and (iii) a bundle contained in $\emptyset$ is less preferred.[13]

Hence the domain of essentially monotonic preferences includes not only all monotonic preferences, but also preferences like the one below, where the critical bundles are underlined.

$$P'_i: \quad \underline{apb} \succ \underline{ap} \succ ab \succ a \succ \underline{\emptyset} \succ b \succ p \succ pb$$

An agent having the above preference treats a bundle acceptable (better than getting nothing) if and only if it contains an apartment. Moreover, she treats the bicycle slots and parking slots as benefits if she gets an apartment. However, she treats these as bads otherwise.[14] The classical monotonicity requirement captures only complete complementarity: the more the better no matter the status quo. As shown by P_i, getting additional objects is always preferred. However, our notion of essential monotonicity captures also "partial complementarity." As shown by P'_i, getting an apartment is always preferred. But whether getting a bicycle slot and a parking slot is preferred depends on whether the agent already gets an apartment.

Example 11. *Now consider the following alternative scenario, where we seek to allocate three types of tasks, a, b, and c respectively, among a given set of agents. In particular, assume that* 50 *working days of task a,* 50 *working days of task b, and* 30 *working days of task c have to be allocated among 100 team members.* ■

In this situation, objects appear to be "bads" and it would appear that the essential monotonicity requirement is not applicable to such situations as it would require, in the very least, that the grand bundle is the favorite. But the following preference, which seems to be reasonable in this situation, treats the empty set as the best.

$$P'': \emptyset \succ a \succ b \succ c \succ ab \succ ac \succ bc \succ abc$$

However, the following observation makes essential monotonicity applicable. We need only to treat the following as the objects to be allocated: 50 copies of "not serving a working day of task a," 50 copies of "not serving a working day of task b," and 70 copies of "not serving a working day of task c." These imaginary objects are denoted as $\bar{a}$, $\bar{b}$, and $\bar{c}$ respectively. For this new problem, the critical bundles are $\bar{a}\bar{b}\bar{c}$, $\bar{c}$, and $\emptyset$. Then the preference P'' above can be translated to $\bar{P}''$, which is essentially monotonic for the imaginary problem.

$$P'': \underline{\bar{a}\bar{b}\bar{c}} \succ \bar{b}\bar{c} \succ \bar{a}\bar{c} \succ \bar{a}\bar{b} \succ \underline{\bar{c}} \succ \bar{b} \succ \bar{a} \succ \underline{\emptyset}$$

[13]Notice that the last restriction is vacuous. However, we still present it as a restriction in order to simplify definition.

[14]Put otherwise, receiving a parking slot or a bicycle slot without living nearby is costly. This is plausible for two reasons. Firstly, objects obtained from a publicly financed project are usually not allowed to be used for profit; it would accordingly be very difficult to benefit from renting these out. Secondly, receiving a parking slot in a project may possibly exclude her from getting a parking slot in future projects where she may be assigned an apartment.

Hence, once these imaginary objects are allocated, the tasks are automatically allocated. Moreover, essential monotonicity should be a reasonable preference restriction for these imaginary objects.

4.2 Examples

Example 12. RSDB rule is not sd-efficient on the universal domain.

This example is a modification of an example in Bogomolnaia and Moulin (2001). Let $I = \{1,2,3,4\}$, $X = \{a,b,c\}$, and $q_a = q_b = q_c = 1$. Consider the preference profile P given below.

$$\begin{array}{lcccc} P_1, P_2: & ab & c & \emptyset & \cdots \\ P_3, P_4: & c & ab & \emptyset & \cdots \end{array}$$

Then the random assignment specified by the RSDB rule is given below, where B denotes an arbitrary bundle different from ab, c, and $\emptyset$. The reader can verify that it is not sd-efficient because it is dominated by the random assignment L.

$$RSDB(P) = \begin{pmatrix} & ab & c & \emptyset & B \\ 1,2: & 5/12 & 1/12 & 1/2 & 0 \\ 3,4: & 1/12 & 5/12 & 1/2 & 0 \end{pmatrix} \quad L = \begin{pmatrix} & ab & c & \emptyset & B \\ 1,2: & 1/2 & 0 & 1/2 & 0 \\ 3,4: & 0 & 1/2 & 1/2 & 0 \end{pmatrix} \quad \blacksquare$$

Example 13. PSB rule is not sd-strategy-proof on the universal domain.

Let $I = \{1,2,3,4\}$, $X = \{a,b\}$, and $q_a = q_b = 1$. Two preferences are as below.

$$\begin{array}{lccccccc} \tilde{P}_i: & ab & \succ & a & \succ & b & \succ & \emptyset \\ \hat{P}_i: & a & \succ & ab & \succ & b & \succ & \emptyset \end{array}$$

Let two preference profiles be $P = (\tilde{P}_1, \tilde{P}_2, \hat{P}_3, \hat{P}_4)$ and $P' = (\hat{P}_1, \tilde{P}_2, \hat{P}_3, \hat{P}_4)$. The following are the corresponding assignments specified by the PSB rule. In particular, $L = PSB(P)$ and $L' = PSB(P')$.

	ab	a	b	$\emptyset$		ab	a	b	$\emptyset$
L_1 :	1/4	0	1/8	5/8	L'_1 :	0	1/4	3/16	9/16
L_2 :	1/4	0	1/8	5/8	L'_2 :	1/4	0	3/16	9/16
L_3 :	0	1/4	1/8	5/8	L'_3 :	0	1/4	3/16	9/16
L_4 :	0	1/4	1/8	5/8	L'_4 :	0	1/4	3/16	9/16

Across the two preference profiles, agent 1 is the unique deviator. Notice that $L'_{1ab} + L'_{1a} + L'_{1b} = 7/16 > 6/16 = L_{1ab} + L_{1a} + L_{1b}$, which means that, by misreporting P'_1, agent 1 receives a higher probability of getting a bundle better than $\emptyset$. Hence the PSB rule is manipulable on any domain containing these two preferences, including the universal domain. $\blacksquare$

Example 14. PSB rule is not sd-strategy-proof on the sequentially dichotomous domain.

In the setting with objects, Liu (2019) proved that the PS rule is sd-strategy-proof on the sequentially dichotomous domain, which is generated by lexicographically checking a fixed list of properties. We first introduce the sequentially dichotomous domain to the current setting.

Let $x_1, x_2, \cdots, x_m$ be a fixed ordering of objects. To simplify notation, we denote a bundle as a 0-1 vector of length m, where a "one" at the t-th position means this bundle contains x_t and a "zero" means not. For example $A = (1,0,0,1)$ is equivalent to $A = x_1x_4$. For each bundle A and each index $t = 1, \cdots, m$, we write A_t as its t-th element and A^t the sequence of the first t elements. For example, for the bundle $A = (1,0,0,1)$, $A_2 = 0$ and $A^3 = (1,0,0)$. A preference is sequentially dichotomous if the bundles are ranked in the following sequential way. First, either every bundle containing x_1 is better than every bundle that does not, or the other way around. Next, within the bundles containing x_1, either every bundle containing x_2 is better than every bundle that does not, or the other way around. Similarly, within the bundles that do not contain x_1, either every one containing x_2 is better than every one that does not, or the other way around. The preference is refined by checking sequentially for x_3, x_4, and so on.

Formally, a preference $P_i \in \mathbb{P}$ is sequentially dichotomous if

1. either $[\forall A, B \in \mathcal{X}$ s.t. $A_1 = 1, B_1 = 0, A \; P_i \; B]$ or $[\forall A, B \in \mathcal{X}$ s.t. $A_1 = 1, B_1 = 0, B \; P_i \; A]$;

2. $\forall t = 2, \cdots, m$ and $\forall \alpha \in \{0,1\}^{t-1}$, either $[\forall A, B \in \mathcal{X}$ s.t. $A^{t-1} = B^{t-1} = \alpha, A_t = 1, B_t = 0, A \; P_i \; B]$ or $[\forall A, B \in \mathcal{X}$ s.t. $A^{t-1} = B^{t-1} = \alpha, A_t = 1, B_t = 0, B \; P_i \; A]$.

In this manner, the preference structure of the sequentially dichotomous domain of Liu (2019) is directly introduced into the bundle setting.[15] The preferences $\tilde{P}_i$ and $\hat{P}_i$ in Example 13 are instances of sequentially dichotomous preferences. In particular, agents compare the bundles by checking first whether the bundle contains a and second whether it contains b. Both preferences prefer the bundles containing a (ab and a) to the bundles that do not (b and $\emptyset$). Then between ab and a, $\tilde{P}_i$ prefers the one containing b while $\hat{P}_i$ prefers the one that does not. Between b and $\emptyset$, both preferences prefer the one containing b.

The manipulation of the PSB rule in Example 13 indicates that the possibility result on the sequentially dichotomous domain in the classical random assignment model fails in the setting with bundles. This failure occurs exactly because the definition of feasibility is modified. If we now treat the four bundles as distinct objects, feasibility of random assignments of individual objects now dictates that every column sums to one. Then the random assignments generated by the PS rule for the above profiles would be as follows.

	ab	a	b	$\emptyset$		ab	a	b	$\emptyset$
$L_1:$	$1/2$	0	$1/4$	$1/4$	$L'_1:$	$1/6$	$1/3$	$1/4$	$1/4$
$L_2:$	$1/2$	0	$1/4$	$1/4$	$L'_2:$	$1/2$	0	$1/4$	$1/4$
$L_3:$	0	$1/2$	$1/4$	$1/4$	$L'_3:$	$1/6$	$1/3$	$1/4$	$1/4$
$L_4:$	0	$1/2$	$1/4$	$1/4$	$L'_4:$	$1/6$	$1/3$	$1/4$	$1/4$

[15] By treating "containing x_t" as the t-th property, the above given definition is equivalent to the original one in Liu (2019).

It is evident that agent 1's misreport is no longer profitable. To summarize, it follows that the change in the feasibility requirement from the classical setting to the setting with bundles has a significant implication on possibilities of designing a desirable rule, in that a previously known possibility result fails. ■

Example 15. PSB rule is not decomposable on the universal domain.

Let $I = \{1, 2, 3\}$, $X = \{a, b, c\}$, and $q_x = 1\ \forall x \in X$. Let the preference profile P and the random assignment $L = PSB(P)$ be as below.

$$\begin{array}{llllll} P_1: & ab & c & b & \emptyset & \cdots \\ P_2: & c & b & a & \emptyset & \cdots \\ P_3: & c & a & b & \emptyset & \cdots \end{array} \qquad \begin{array}{cccccc} & ab & a & b & c & \emptyset \\ L_1: & 3/4 & 0 & 0 & 0 & 1/4 \\ L_2: & 0 & 0 & 1/4 & 1/2 & 1/4 \\ L_3: & 0 & 1/4 & 0 & 1/2 & 1/4 \end{array}$$

We prove that L is not decomposable. Suppose otherwise and let $L = \sum_{D \in \mathcal{D}} \beta(D) \cdot D$. Let in addition D^1 and D^2 be two deterministic assignments such that $D^1_{1ab} = D^1_{2c} = D^1_{3\emptyset} = 1$ and $D^2_{1ab} = D^2_{2\emptyset} = D^2_{3c} = 1$. Since D^1 and D^2 are the only ones where agent 1 receives ab, $3/4 = \beta(D^1) + \beta(D^2)$. Moreover, since at D^1, agent 3 receives nothing and $L_{3\emptyset} = 1/4$, $1/4 \geqslant \beta(D^1)$. Similarly, $L_{2\emptyset} = 1/4 \geqslant \beta(D^2)$. Hence, we have a contradiction: $1/4 + 1/4 \geqslant 3/4$. ■

Example 16. The essentially monotonic domain is not maximal for the equivalence between the RSDB rule and the PSB rule.

Let $X = \{a, b\}$, $I = \{1, 2, 3\}$, $q_a = 2$, and $q_b = 1$. The critical bundles and capacities are $A_1 = ab$, $d_1 = 1$, $A_2 = a$, $d_2 = 1$, $A_3 = \emptyset$, and $d_3 = 1$. Then the essentially monotonic domain, $\mathbb{D}_{EM}$, contains only the following three preferences

$$\begin{array}{lllll} P_i: & ab & b & a & \emptyset \\ \hat{P}_i: & ab & a & b & \emptyset \\ \tilde{P}_i: & ab & a & \emptyset & b \end{array}$$

Consider now a preference $\bar{P}_i$ below

$$\begin{array}{lllll} \bar{P}_i: & ab & \emptyset & a & b \end{array}$$

One can verify that $PSB(P) = RSDB(P)$ for all $P \in \{\mathbb{D}_{EM} \cup \bar{P}_i\}^n$. ■

4.3 Final Remarks

We study the random assignments of bundles with no free disposal. The induced feasibility requirement has been shown to have significant implications for the design of random assignment rules. First, the characterization of sd-efficiency is fundamentally different in this setup. Second, the possibility result of Liu (2019) fails under this new feasibility requirement. However, we identify a preference domain on which a desirable existence result exists by showing that the PSB rule is equivalent to the RSDB rule. In this regard, we observe that the essentially monotonic domain is not maximal for this equivalence. This is illustrated in Example 16. It could be interesting to investigate whether there are situations where the essentially monotonic domain is maximal for the equivalence, and what the maximal domains in general are.

Two unresolved problems are presumably of interest. The first is whether there exists a non-constant desirable rule on the essentially monotonic domain, and if so, can it be eliminated by imposing other axioms like non-bossiness.[16] The second question is whether there exist other preference domains which admit a desirable rule.

5 Proofs

For clarity of the presentation, we begin with the proof of Proposition 1. Thereafter, we present in order the proofs of Properties 1, 2, 3, and the proof of Proposition 2.

5.1 Proof of Proposition 1

Necessity: We show the contrapositive statement. Let $L \in \mathcal{L}$ be balanced at $P \in \mathbb{P}^n$. Then there is an $\alpha : \mathcal{T} \to \mathbb{R}_+$ such that (i) $\alpha(i, A, B) > 0$ implies $L_{iA} > 0$ and $B\ P_i\ A$, (ii) $\forall x \in X : \sum_{\{(i,A,B)\in\mathcal{T}:x\in B\}} \alpha(i, A, B) = \sum_{\{(i,A,B)\in\mathcal{T}:x\in A\}} \alpha(i, A, B)$. We show L is not sd-efficient at P. To do this, we construct another matrix L'. Let $\epsilon \in \mathbb{R}_{++}$ be a very small positive number and let for $\forall j \in I$ and $\forall C \in \mathcal{X}$,

$$L'_{jC} = L_{jC} + \sum_{\{(i,A,B)\in\mathcal{T}:i=j,B=C\}} \epsilon \cdot \alpha(i, A, B) - \sum_{\{(i,A,B)\in\mathcal{T}:i=j,A=C\}} \epsilon \cdot \alpha(i, A, B).$$

Notice first that by letting ϵ to be sufficiently small, we can guarantee $L'_{jC} \geqslant 0$ for all $j \in I$ and $C \in \mathcal{X}$. Then the following two classes of equations show that L' is a feasible random assignment, in other words $L' \in \mathcal{L}$.

$$\begin{aligned}
\forall j \in I : \sum_{C\in\mathcal{X}} L'_{jC} &= \sum_{C\in\mathcal{X}} L_{jC} + \sum_{C\in\mathcal{X}} \sum_{\{(i,A,B)\in\mathcal{T}:i=j,B=C\}} \epsilon \cdot \alpha(i, A, B) \\
&\quad - \sum_{C\in\mathcal{X}} \sum_{\{(i,A,B)\in\mathcal{T}:i=j,A=C\}} \epsilon \cdot \alpha(i, A, B) \\
&= 1 + \sum_{\{(i,A,B)\in\mathcal{T}(P,L):i=j\}} \epsilon \cdot [\alpha(i, A, B) - \alpha(i, A, B)] = 1 \\
\forall x \in X : \sum_{i\in I,x\in C} L'_{jC} &= \sum_{i\in I,x\in C} L_{jC} + \sum_{\{(i,A,B)\in\mathcal{T}:x\in B\}} \epsilon \cdot \alpha(i, A, B) \\
&\quad - \sum_{\{(i,A,B)\in\mathcal{T}:x\in A\}} \epsilon \cdot \alpha(i, A, B) \\
&= \sum_{i\in I,x\in C} L_{jC} = q_x.
\end{aligned}$$

The last equality follows from the definition of α. Next, it is evident that $L'_j\ P^{sd}_j\ L_j$ for all $j \in I$ since the probability transfers are all from less preferred bundles to more preferred ones. Consequently, L is dominated by L' constructed above and hence not sd-efficient at P.

[16] We thank an anonymous referee for suggesting this.

Sufficiency: We show the contrapositive statement. Let L be sd-inefficient at P. Then there is another random assignment $L' \in \mathcal{L}$ such that $L' \neq L$ and $L'_j \ P^{sd}_j \ L_j$ for all j. We construct an $\alpha : \mathcal{T}(P, L) \to \mathbb{R}_+$ such that (i) $\alpha(i, A, B) > 0$ implies $L_{iA} > 0$ and $B \ P_i \ A$, (ii) $\forall x \in X$: $\sum_{\{(i,A,B)\in\mathcal{T}:x\in B\}} \alpha(i, A, B) = \sum_{\{(i,A,B)\in\mathcal{T}:x\in A\}} \alpha(i, A, B)$.

By the fact that $L_j, L'_j \in \Delta(\mathcal{X})$ for all $j \in I$, there is a system of probability transfers $\beta : \mathcal{T} \to \mathbb{R}_+$ such that, $\forall j \in I$ and $C \in \mathcal{X}$,

$$L'_{jC} = L_{jC} + \sum_{\{(i,A,B)\in\mathcal{T}:i=j,B=C\}} \beta(i, A, B) - \sum_{\{(i,A,B)\in\mathcal{T}:i=j,A=C\}} \beta(i, A, B). \tag{1}$$

In other words, L' is constructed from L by β. In addition, since both L and L' are feasible random assignments, $\forall x \in X$,

$$\sum_{\{(i,A,B)\in\mathcal{T}:x\in B\}} \beta(i, A, B) = \sum_{\{(i,A,B)\in\mathcal{T}:x\in A\}} \beta(i, A, B). \tag{2}$$

Note that the vector β in general can not serve as the α we seek since β may transfer some positive probability from (i, A) to (i, B) where $L_{iA} = 0$ and(or) $A \ P_i \ B$, which is not allowed by the definition of α. In the following, we construct the sought α from β in two steps.

Step 1: Given $\beta : \mathcal{T} \to \mathbb{R}_+$ satisfying (1) and (2), we construct a $\gamma : \mathcal{T} \to \mathbb{R}_+$ satisfying not only (1) and (2) but also that $\gamma(i, A, B) > 0$ implies $L_{iA} > 0$.

To do so, pick an arbitrary $(i, A, B) \in \mathcal{T}$ such that $\beta(i, A, B) > 0$ and $L_{iA} = 0$. We claim that there is another bundle $C \neq A$ such that $\beta(i, C, A) > 0$, since otherwise, according to (1), we have a contradiction: $L'_{iA} \leqslant 0 + 0 - \beta(i, A, B) < 0$.

Let $\beta(i, A, B) = u$ and $\beta(i, C, A) = v$. Then we update β to β' by the following changes.

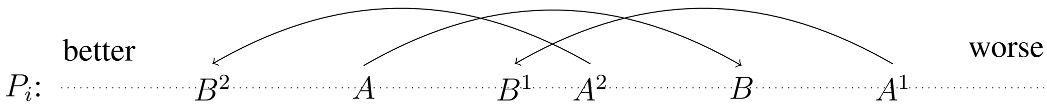
$$\text{If } u \leqslant v, \text{ let } \beta'(i, A, B) = 0, \beta'(i, C, B) = \beta(i, C, B) + u, \beta'(i, C, A) = v - u;$$

$$\text{If } u > v, \text{ let } \beta'(i, A, B) = u - v, \beta'(i, C, B) = \beta(i, C, B) + v, \beta'(i, C, A) = 0.$$

Notice first that no matter whether $u \leqslant v$ or not, β' still satisfies (1) and (2). Notice in addition that for the case where $u \leqslant v$, $\beta'(i, A, B) = 0$ and hence the unwanted instance where $\beta(i, A, B) > 0$ and $L_{iA} = 0$ is eliminated, while for the other case where $u > v$, this unwanted instance remains. We can then repeat the update above by finding some other bundle $D \neq A$ such that $\beta'(i, D, A) > 0$. By repeatedly applying the above update, we can finally construct a vector $\gamma : \mathcal{T} \to \mathbb{R}_+$ which satisfies not only (1) and (2) but also that $\gamma(i, A, B) > 0$ implies $L_{iA} > 0$.

Step 2: Given $\gamma : \mathcal{T} \to \mathbb{R}_+$ generated by the last step, we construct the sought $\alpha : \mathcal{T} \to \mathbb{R}_+$ that satisfies not only (1) and (2), but also the property that $\alpha(i, A, B) > 0$ implies $L_{iA} > 0$ and $B \ P_i \ A$. In other words, positive probability transfers are allowed only from less preferred bundles to preferred bundles.

To do so, pick an arbitrary $(i, A, B) \in \mathcal{T}$ such that $\gamma(i, A, B) > 0$ and $A \ P_i \ B$. We claim that there is a sequence $(i, A^l, B^l)_{l=1}^L \subset \mathcal{T}$ such that (i) $\gamma(i, A^l, B^l) > 0$ and $B^l \ P_i \ A^l$ for all $l = 1, \cdots, L$, (ii) $B^1 \ R_i \ B \ R_i \ A^1$, (iii) $B^l \ R_i \ A^{l+1}$ for all $l = 1, \cdots, L - 1$, and (iv) $B^L \ R_i \ A \ R_i \ A^L$. The following figure depicts an instance where $L = 2$.

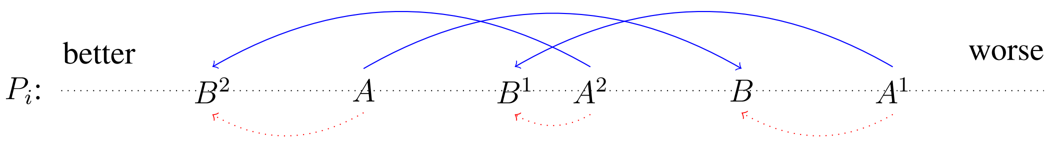


We show the existence of such a sequence by construction. First, notice that there exists $(i, A^1, B^1) \in \mathcal{T}$ such that $\gamma(i, A^1, B^1) > 0$, $B^1\ P_i\ A^1$, and $B^1\ R_i\ B\ R_i\ A^1$, since otherwise, $\sum_{CP_iB} L'_{iC} < \sum_{CP_iB} L_{iC}$ which contradicts $L'_i\ P^{sd}_i\ L_i$. Fixing A^1 and B^1, if $B^1\ R_i\ A$, we are done by letting $L = 1$. If not, there exists another triple $(i, A^2, B^2) \in \mathcal{T}$ such that $\gamma(i, A^2, B^2) > 0$, $B^2\ P_i\ A^2$, and $B^1\ R_i\ A^2$, since otherwise, $\sum_{CP_iB^1} L'_{iC} < \sum_{CP_iB^1} L_{iC}$ which contradicts $L'_i\ P^{sd}_i\ L_i$. We repeat this procedure to identify the sequence, and finally the finiteness of bundles gives $B^L\ R_i\ A\ R_i\ A^L$.

Fixing such a sequence, let $\mu = \min\{\gamma(i, A, B), \gamma(i, A^l, B^l) : l = 1, \cdots, L\}$. Then we update γ to γ' using exactly the following changes:

$$\begin{aligned}
\gamma'(i, A, B) &= \gamma(i, A, B) - \mu \\
\gamma'(i, A^l, B^l) &= \gamma(i, A^l, B^l) - \mu, \quad \forall l = 1, \cdots, L \\
\gamma'(i, A, B^L) &= \gamma(i, A, B^L) + \mu \\
\gamma'(i, A^1, B) &= \gamma(i, A^1, B) + \mu \\
\gamma'(i, A^{l+1}, B^l) &= \gamma(i, A^{l+1}, B^l) + \mu, \quad \forall l = 1, \cdots, L-1
\end{aligned}$$

The following figure depicts the update, where blue solid arrows corresponds to the minuses and the red dotted arrows the pluses.



It is evident that $\gamma : \mathcal{T}(L) \to \mathbb{R}_+$ satisfies not only (1) and (2) but also that $\gamma'(i, A, B) > 0$ implies $L_{iA} > 0$. It satisfies (1) evidently. To see that it satisfies (2), notice that for all $C \in \mathcal{X}$, $\sum_{\{(i,A,B)\in\mathcal{T}:A=C\}} \gamma'(i, A, B) = \sum_{\{(i,A,B)\in\mathcal{T}:A=C\}} \gamma(i, A, B)$ and $\sum_{\{(i,A,B)\in\mathcal{T}:B=C\}} \gamma'(i, A, B) = \sum_{\{(i,A,B)\in\mathcal{T}:B=C\}} \gamma(i, A, B)$.

By repeatedly applying the above update, we can finally construct an $\alpha : \mathcal{T} \to \mathbb{R}_+$ satisfying not only (1) and (2) but also that $\alpha(i, A, B) > 0$ implies $L_{iA} > 0$ and $B\ P_i\ A$.

To show the sufficiency part, it remains to show that such an α satisfies$\forall x \in X$:

$$\sum_{\{(i,A,B)\in\mathcal{T}:x\in B\}} \alpha(i, A, B) = \sum_{\{(i,A,B)\in\mathcal{T}:x\in A\}} \alpha(i, A, B),$$

which follows from the fact that α satisfies (2).

5.2 Proof of Property 1

First, by construction the RSDB rule is decomposable, because it is a lottery over deterministic assignment rules. Second, the RSDB rule is sd-strategy-proof, because each SDB rule is sd-strategy-proof and that sd-strategy-proofness is preserved under linear combinations. Third, the RSDB rule treats equals equally, because the various orderings of agents are equally weighted. Last, the RSDB rule is not sd-efficient because it selects a random assignment that is not sd-efficient at some preference profile (see Example 12 in Section 4).

5.3 Proof of Property 2

Examples 13 and 15 in Section 4 show respectively that the PSB rule is not sd-strategy-proof nor decomposable on the universal domain. The fact that the PSB rule treats equals equally follows from the fact that all the agents have the same eating speed at every point in time. Now we turn to proving the efficiency claim. To do so, we introduce two conditions on random assignments. One is called the strong unbalancedness, which is a sufficient condition for sd-efficiency. The other is called the weak unbalancedness and it is necessary for sd-efficiency. The relation between these conditions is summarized in Lemma 1. We then show that the PSB rule always generates strongly unbalanced random assignments, which hence guarantees sd-efficiency of the PSB rule. Although, for the current purpose, only a part of Lemma 1 is needed, we nonetheless present the lemma in its entirety in the hope that it may be useful for other related studies.

In order to define these two conditions, we introduce some notation. Fix $P \in \mathbb{P}^n$, $L \in \mathcal{L}$, and a subset of triples $\mathcal{S} \subset \mathcal{T}$. For each object $x \in X$, let $d(x, \mathcal{S})$ count the triples $(i, A, B) \in \mathcal{S}$ such that $x \in B$ and $x \notin A$. Recall that we interpret a triple (i, A, B) as a potential transfer from (i, A) to (i, B). Then for every such x, there is a positive influence on the feasibility of x when a positive transfer is implemented. (See Example 8.) In this way, $d(x, \mathcal{S})$ counts the instances of positive influence on the feasibility of x. Similarly, $s(x, \mathcal{S})$ counts the instances of negative influence on the feasibility of x. Formally

$$d(x, \mathcal{S}) \equiv \big|\{(i, A, B) \in \mathcal{S} : x \in B \backslash A\}\big| \text{ and } s(x, \mathcal{S}) \equiv \big|\{(i, A, B) \in \mathcal{S} : x \in A \backslash B\}\big|.$$

We say an assignment $L \in \mathcal{L}$ is **strongly unbalanced** at $P \in \mathbb{P}^n$ if there is no $\mathcal{S} \subset \mathcal{T}$ s.t.

(i) $\forall\ (i, A, B) \in \mathcal{T}$: $(i, A, B) \in \mathcal{S}$ implies $L_{iA} > 0$ and $B\ P_i\ A$,

(ii) $\forall x \in X$: $d(x, \mathcal{S}) > 0 \Leftrightarrow s(x, \mathcal{S}) > 0$.

We say L is **strongly balanced** at P if it is not strongly unbalanced at P.

Next, an assignment $L \in \mathcal{L}$ is **weakly unbalanced** at $P \in \mathbb{P}^n$ if there is no $\mathcal{S} \subset \mathcal{T}$ s.t.

(i) $\forall\ (i, A, B) \in \mathcal{T}$: $(i, A, B) \in \mathcal{S}$ implies $L_{iA} > 0$ and $B\ P_i\ A$,

(ii) $\forall x \in X$: $d(x, \mathcal{S}) = s(x, \mathcal{S})$.

We say L is **weakly balanced** at P if it is not weakly unbalanced at P.

The following lemma establishes the logical relations among the definitions we have mentioned so far. The proof is in the Appendix.

Lemma 1. *Strong unbalancedness* $\substack{\Rightarrow \\ \not\Leftarrow}$ *sd-efficiency* $\substack{\Rightarrow \\ \not\Leftarrow}$ *weak unbalancedness* $\substack{\Rightarrow \\ \not\Leftarrow}$ *acyclicity.*

Let $P \in \mathbb{P}^n$ and $L = PSB(P)$. We show that L is sd-efficient at P.

To do so, $\forall\ x \in X$, let $t(x)$ be the time when x is depleted, i.e., $t(x) \equiv \min\{t^v : r^v_x \leqslant 0\}$. Similarly, $\forall\ x \in X$, let $t(\bar{x})$ be the time when $\bar{x}$ is depleted, i.e., $t(\bar{x}) \equiv \min\{t^v : r^v_{\bar{x}} \leqslant 0\}$. We consider two cases.

Case 1: $\forall x \in X$, $t(x) \leqslant t(\bar{x})$. Lemma 2 below considers this case.

Lemma 2. *Let* $P \in \mathbb{P}^n$ *and* $L = PSB(P)$*. If,* $\forall x \in X$*,* $t(x) \leqslant t(\bar{x})$*,* L *is sd-efficient at* P*.*

Proof. We prove the lemma by contradiction. Suppose L is not sd-efficient at P. Then by Lemma 1, L is strongly balanced at P. Put otherwise, there is a subset $\mathcal{S} \subset \mathcal{T}$ such that (i) $(i, A, B) \in \mathcal{S}$ implies $B\ P_i\ A$ and $L_{iA} > 0$; (ii) $\forall x \in X$, $[\exists (i, A, B) \in \mathcal{S}$ s.t. $x \in A \backslash B] \Leftrightarrow [\exists (i, A, B) \in \mathcal{S}$ s.t. $x \in B \backslash A]$. For each $i \in I$ and $A \in \mathcal{X}$ such that $L_{iA} > 0$, let $t(i, A)$ denote the time when agent i starts to consume A. Formally, $t(i, A) \equiv \min\{t(x) : x \in A\} - L_{iA}$.

Pick an arbitrary $(i_1, A_1, B_1) \in \mathcal{S}$, by definition, $B_1\ P_{i_1}\ A_1$ and $L_{i_1 A_1} > 0$. Hence at the time when agent i_1 starts to consume A_1, i.e., $t(i_1, A_1)$, B_1 is already depleted. Then the assumption that, $\forall\ x \in X$, $t(x) \leqslant t(\bar{x})$, implies the existence of $x_1 \in B_1 \backslash A_1$ such that $t(x_1) \leqslant t(i_1, A_1)$. Strong balancedness then implies the existence of $(i_2, A_2, B_2) \in \mathcal{S}$ such that $x_1 \in A_2 \backslash B_2$. Then $L_{i_2 A_2} > 0$ implies $t(i_2, A_2) < t(x_1)$. Similarly, let $x_2 \in B_2 \backslash A_2$ be arbitrary such that $t(x_2) \leqslant t(i_2, A_2)$. Hence $t(x_2) \leqslant t(i_2, A_2) < t(x_1)$. We repeat the procedure to find x_3, A_3, and i_3 such that $t(x_3) \leqslant t(i_3, A_3) < t(x_2) \leqslant t(i_2, A_2) < t(x_1)$. If $x_3 = x_1$, we have a contradiction. Otherwise, we repeat the procedure to find x_4, and so on. Finally, the finiteness of X implies the existence of x such that $t(x) < t(x)$: contradiction. ■

Case 2: Let $\bar{X} \equiv \{x \in X : t(x) > t(\bar{x})\}$ be nonempty. Let $\mathcal{E} \equiv (I, X, q)$ denote the model setting. We define a new model $\mathcal{E}' \equiv (I, Y, p)$ such that (i) the set of agents is the same as the original model, and (ii) the set of objects Y and their capacities p are associated to X and q via an arbitrary bijection $f : Y \to X$, as follows.

$$p_y = \begin{cases} q_{f(y)}, & f(y) \in X \backslash \bar{X} \\ n - q_{f(y)}, & f(y) \in \bar{X} \end{cases}$$

Thus if an object $y \in Y$ is mapped to an object x not in $\bar{X}$, its capacity is the same as x. Otherwise, its capacity is defined as n minus the capacity of x. For $\mathcal{E}$ and $\mathcal{E}'$, the set of bundles are denoted as $\mathcal{X}$ and $\mathcal{Y}$ respectively. It is evident that $|\mathcal{X}| = |\mathcal{Y}|$. In addition, the set of random assignments are denoted as $\mathcal{L}$ and $\mathcal{L}'$ respectively. We now define a mapping $g : \mathcal{Y} \to \mathcal{X}$ such that $\forall\ A \in \mathcal{Y}$, $g(A) = B \in \mathcal{X}$ if and only if, $\forall y \in Y$,

$$\begin{aligned} & f(y) \in B, \text{ if either } [f(y) \in X \backslash \bar{X} \text{ and } y \in A] \text{ or } [f(y) \in \bar{X} \text{ and } y \notin A] \\ & f(y) \notin B, \text{ otherwise.} \end{aligned} \tag{3}$$

For a better understanding of the construction, we illustrate in Example 17 the construction with a specific model setting. One can verify that g is a bijection. For the new model $\mathcal{E}'$, we specify a profile of preferences on $\mathcal{Y}$, denoted as $P' = (P'_i)_{i \in I}$, and a random assignment

$L' \in \mathcal{L}'$. In particular, for all $i \in I$ and $A, B \in \mathcal{Y}$, $A \ P_i' \ B$ if and only if $g(A) \ P_i \ g(B)$. For all $i \in I$ and $A \in \mathcal{Y}$, $L'_{iA} = L_{ig(A)}$.

We now make the following two claims.

Claim 1: L' is sd-efficient at P' in $\mathcal{E}' \Rightarrow L$ is sd-efficient at P in $\mathcal{E}$.

We prove the contrapositive statement. Let L be not sd-efficient at P in $\mathcal{E}$. Then, by definition, $\exists \, \tilde{L} \in \mathcal{L}$ such that $\tilde{L} \neq L$ and $\tilde{L}_i \ P_i^{sd} \ L_i$ for all $i \in I$. We construct a matrix $\tilde{L}' \in [0,1]^{I \times \mathcal{Y}}$ such that, $\forall \, i \in I$ and $A \in \mathcal{Y}$, $\tilde{L}'_{iA} = \tilde{L}_{ig(A)}$. We prove the following three statements.

1. $\tilde{L}'$ is a random assignment in $\mathcal{E}'$, i.e., $\tilde{L}' \in \mathcal{L}'$.

To see this, note that, $\forall i \in I$,

$$\sum_{A \in \mathcal{Y}} \tilde{L}'_{iA} = \sum_{A \in \mathcal{Y}} \tilde{L}_{ig(A)} = \sum_{A \in \mathcal{X}} \tilde{L}_{iA} = 1.$$

Moreover, $\forall \, y \in Y$,

$$\begin{aligned} f(y) \in X \backslash \bar{X}: \quad \sum_{i \in I, y \in A} \tilde{L}'_{iA} &= \sum_{i \in I, y \in A} \tilde{L}_{ig(A)} && (\text{ by } \tilde{L}'_{iA} = \tilde{L}_{ig(A)}) \\ &= \sum_{i \in I, f(y) \in A} \tilde{L}_{iA} && (\text{ by the definition of } g) \\ &= q_{f(y)} && (\text{ by} \tilde{L} \in \mathcal{L}) \\ &= p_y. && (\text{ by the definition of } p) \end{aligned}$$

$$\begin{aligned} f(y) \in \bar{X}: \quad \sum_{i \in I, y \in A} \tilde{L}'_{iA} &= \sum_{i \in I, y \in A} \tilde{L}_{ig(A)} && (\text{ by } \tilde{L}'_{iA} = \tilde{L}_{ig(A)}) \\ &= \sum_{i \in I, f(y) \notin A} \tilde{L}_{iA} && (\text{ by the definition of } g) \\ &= n - \sum_{i \in I, f(y) \in A} \tilde{L}_{iA} = n - q_{f(y)} && (\text{ by} \tilde{L} \in \mathcal{L}) \\ &= p_y. && (\text{ by the definition of } p) \end{aligned}$$

2. $\tilde{L}' \neq L'$. This is implied by the fact that $\tilde{L} \neq L$ and that g is a bijection.

3. $\forall \, i \in I$, $\tilde{L}'_i \ P_i'^{sd} \ L'_i$. Given $\forall \, i \in I$, $B \in \mathcal{Y}$, and the fact that $\tilde{L}_i \ P_i^{sd} \ L_i$,

$$\sum_{\{A \in \mathcal{Y}: AR'_i B\}} \tilde{L}'_{iA} - \sum_{\{A \in \mathcal{Y}: AR'_i B\}} L'_{iA} = \sum_{\{g(A) \in \mathcal{X}: g(A) R_i g(B)\}} \tilde{L}_{iA} - \sum_{\{g(A) \in \mathcal{X}: g(A) R_i g(B)\}} L_{iA} \geqslant 0.$$

The above three statements together imply that L' is not sd-efficient at P' in $\mathcal{E}'$.

Claim 2: For $\mathcal{E}'$, $L' = PSB(P')$ and $t(y) \leqslant t(\bar{y})$ for all $y \in Y$.

By construction, when the PSB rule is applied to P' in $\mathcal{E}'$, if $f(y) \in X \backslash \bar{X}$, y mimics $f(y)$ when the PSB rule is applied to P in $\mathcal{E}$. If instead $f(y) \in \bar{X}$, $\bar{y}$ mimics $f(y)$ when the PSB rule is applied to P in $\mathcal{E}$. Hence, $\forall \, i \in I$, $y \in Y$ such that $f(y) \in X \backslash \bar{X}$, and any point in time, agent i consumes y when the PSB rule is applied to P' in $\mathcal{E}'$ if and only if agent i consumes $f(y)$

when the PSB rule is applied to P in $\mathcal{E}$. So $t'(y) = t(x) \leqslant t(\bar{x}) = t'(\bar{y})$, where $t'(y)$ denote the point in time when object y is depleted and $x \in X$ such that $f(y) = x$. On the contrary, $\forall$ $i \in I$, $y \in Y$ such that $f(y) \in \bar{X}$, and any point in time, agent i consumes y when the PSB rule is applied to P' in $\mathcal{E}'$ if and only if agent i consumes $\bar{x}$ when the PSB rule is applied to P in $\mathcal{E}$, where $x = f(y)$. So $t'(y) = t(\bar{x}) \leqslant t(x) = t'(\bar{y})$, where $x \in X$ such that $f(y) = x$.

The statement that $L = PSB(P)$ is sd-efficient at P for Case 2 is now implied by Claims 1, 2 and Lemma 2. In particular, Claim 2 and Lemma 2 imply that L' is sd-efficient at P' in $\mathcal{E}'$. Then Claim 1 implies that L is sd-efficient at P in $\mathcal{E}$. ■

Example 17. *Consider the setting of Example 5. In particular, the model is denoted as $\mathcal{E} = (I, X, q)$, where $I = \{1, 2, 3\}$, $X = \{a, b, c\}$, and $q_a = q_b = q_c = 1$. According to the eating procedure illustrated in Example 5, $t(a) = t(\bar{a}) = 1$, $t(b) = t(\bar{b}) = 1$, $t(\bar{c}) = 2/3 < 1 = t(c)$. Hence, let $\bar{X} = \{c\}$. Let in addition, $\mathcal{E}' = (I, Y, p)$ be the new model where $Y = \{x, y, z\}$, $p_x = q_a = 1$, $p_y = q_b = 1$, and $p_z = 3 - q_c = 2$. Let $f : \{x, y, z\} \to \{a, b, c\}$ be such that $f(x) = a$, $f(y) = b$, and $f(z) = c$. Given this, we map the bundles in $\mathcal{E}'$ to the ones in $\mathcal{E}$ according to the definition of g in (3).*

$$\begin{array}{ccccccccc} & xy & xyz & x & y & xz & yz & \emptyset & z \\ g: & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & abc & ab & ac & bc & a & b & c & \emptyset \end{array}$$

Given the preference profile P and the PSB assignment L in $\mathcal{E}$, we construct the profile P' and a matrix L' in $\mathcal{E}'$ via g.

$$\begin{array}{lcccc} P'_1: & xyz & xy & \cdots & \cdots \\ P'_2: & z & xyz & \emptyset & \cdots \\ P'_3: & z & xyz & \emptyset & \cdots \end{array} \qquad \begin{array}{lcccc} & z & xyz & xy & \emptyset \\ L'_1: & 0 & 2/3 & 1/3 & 0 \\ L'_2: & 2/3 & 0 & 0 & 1/3 \\ L'_3: & 2/3 & 0 & 0 & 1/3 \end{array}$$

One can easily verify that L' is exactly the random assignment generated by applying the PSB rule to P'. One can also verify that in the corresponding eating procedure, each of the objects, x, y, and z in particular, are depleted before their respective opposite objects. ■

5.4 Proof of Property 3

Let $P \in \mathbb{D}^n_{EM}$ be an arbitrary profile of essentially monotonic preferences. We show $\forall i \in I$, $RSDB_i(P) = PSB_i(P) = L_i$ (given below), where $A_1, ..., A_K$ are the critical bundles and B denotes any bundle that is not critical.

$$\begin{array}{lcccccc} & A_1 & A_2 & \cdots & A_{K-1} & A_K & B \\ L_i: & \frac{d_1}{n} & \frac{d_2}{n} & \cdots & \frac{d_{K-1}}{n} & \frac{d_K}{n} & 0 \end{array}$$

The fact that $PSB_i(P) = L_i$ is seen as follows: the eating procedure begins with all agents eating their commonly top ranked bundle, $A_1 = X$, which is exhausted at d_1/n. Thereafter, essential monotonicity implies that the favorite bundle of each agent, within the available bundles, is A_2. So all agents start eating A_2, which is exhausted at $d_1/n + d_2/n$. The iteration goes on until $d_1/n + \cdots + d_{K-1}/n$, when all the objects are depleted. Finally, every agent receives $1 - (d_1/n + \cdots + d_{K-1}/n)$ probability on $\emptyset$, which is equivalent to specifying, for each agent i, $L_{i\emptyset} = d_K/n$.

To see $RSDB_i(P) = L_i$, notice that given an arbitrary ordering σ of agents, $SDB^{\sigma}_{\sigma(i)}(P) = A_1$ for all $i = 1, \cdots, d_1$. Thus each of the first d_1 agents will take their commonly favorite bundle A_1. Thereafter, the agents ordered from $d_1 + 1$ to $d_1 + d_2$ will take A_2 since essential monotonicity implies that this is their favorite bundle among the available ones. This argument continues and gives that $SDB^{\sigma}_{\sigma(i)}(P) = A_k$ for all $i = d_1 + \cdots + d_{k-1} + 1, d_1 + \cdots + d_{k-1} + 2, \cdots, d_1 + \cdots + d_{k-1} + d_k$ and all $k = 2, \cdots, K$. For each $k = 1, \cdots, n$ and each $i \in I$, the probability of this agent occupying the k-th position is $1/n$. To see this, note that there are a total of $n!$ orderings of agents, among which there are $(n-1)!$ orderings where agent i is ordered at the k-th position. Given this, the probability of each agent to get the critical bundle A_1 can be calculated: $\frac{(n-1)!}{n!} \cdot d_1 = d_1/n$. The probabilities of getting other critical bundles can be calculated similarly. Consequently we have $RSDB_i(P) = L_i$.

Note that the random assignment L is determined completely by the critical bundles and critical capacities. In particular, it is independent of the preferences in P. Hence, we have proved that these two rules degenerate to the same constant rule on the essentially monotonic domain. ■

5.5 Proof of Proposition 2

Recall that the number of critical bundles is at most $1 + \max\{q_x : x \in X\}$. In addition, we require $q_x \leqslant n - 1$ for all $x \in X$. Hence, by construction, $n \geqslant K \geqslant 4$. Let $\varphi : \mathbb{P}^n \to \mathcal{L}$ be an sd-strategy-proof, sd-efficient, and sd-envy-free rule. In the following, we construct four preference profiles P^1, P^2, P^3, P^4, and then characterize the random assignments $\varphi(P^1)$, $\varphi(P^2)$, $\varphi(P^3)$, $\varphi(P^4)$. Finally, a contradiction against feasibility is identified, which then proves the theorem.

The preference profiles we construct consist of only the following three preferences

$$\begin{array}{lccccccc} \bar{P}_i: & A_1 & A_3 & A_2 & A_4 & \cdots & A_K & \cdots \\ P_i: & A_1 & A_2 & A_3 & A_4 & \cdots & A_K & \cdots \\ \hat{P}_i: & A_2 & A_1 & A_3 & A_4 & \cdots & A_K & \cdots \end{array}$$

Specifically, the critical bundles are top-ranked and the ranking of A_4 through A_K is the same across the three preferences.

Claim 1: Let the first preference profile be such that all the agents have the same preference as P_i in above table, i.e., $P^1 = (P_1, P_2, P_3, \cdots, P_n)$. Then $\varphi(P^1)$ is as below.

$$\begin{array}{lccccccc} & A_1 & A_2 & A_3 & A_4 & \cdots & A_K & \cdots \\ 1\cdots n: & \frac{d_1}{n} & \frac{d_2}{n} & \frac{d_3}{n} & \frac{d_4}{n} & \cdots & \frac{d_K}{n} & 0 \end{array}$$

By sd-envy-freeness, agents have the same lottery. In order to verify the claim, we notice that there exists $x \in A_1$ such that $q_x = d_1$. Then feasibility of x together with sd-envy-freeness require $L_{iA_1} \leqslant \frac{d_1}{n}$. Similarly, there exists $x \in A_1 \cap A_2$ such that $q_x = d_1 + d_2$, which then implies $L_{iA_1} + L_{iA_2} \leqslant \frac{d_1}{n} + \frac{d_2}{n}$. This argument proceeds until $\sum_{k=1}^{K} L_{iA_k} \leqslant 1$. It is hence evident that any sd-envy-free and feasible assignment $L \neq \varphi(P^1)$ is dominated by $\varphi(P^1)$.

Claim 2: Let $P^2 = (\bar{P}_1, P_2, P_3, \cdots, P_n)$. Then $\varphi(P^2)$ is as follows.

$$\begin{array}{rccccccc} & A_1 & A_2 & A_3 & A_4 & \cdots & A_K & \cdots \\ 1: & \frac{d_1}{n} & 0 & \frac{d_2+d_3}{n} & \frac{d_4}{n} & \cdots & \frac{d_K}{n} & 0 \\ 2\cdots n: & \frac{d_1}{n} & \frac{d_2}{n-1} & \frac{d_3 - \frac{d_2+d_3}{n}}{n-1} & \frac{d_4}{n} & \cdots & \frac{d_K}{n} & 0 \end{array}$$

From P^1 to P^2, agent 1 is the unilateral deviator, and her ranking of A_2 and A_3 are reversed with no other changes. Hence sd-strategy-proofness implies $\varphi_{1A}(P^2) = \varphi_{1A}(P^1)$ for all $A \neq A_2, A_3$. Then sd-envy-freeness implies $\varphi_{iA}(P^2) = \varphi_{1A}(P^2)$ for all $i = 2, \cdots, n$ and all $A \neq A_2, A_3$. Notice that sd-efficiency implies $\varphi_{1A_2}(P^2) = 0$ since otherwise, $\varphi_{iA_3}(P^2) = 0$ for all $i = 2, \cdots, n$, which implies $\varphi_{1A_3}(P^2) = d_3$ and hence $\varphi_{1A_2}(P^2) + \varphi_{1A_3}(P^2) > d_3 \geqslant 1$: a contradiction to feasibility. Given $\varphi_{1A_2}(P^2) = 0$, $\varphi_{1A_3}(P^2) = \frac{d_2+d_3}{n}$ is implied by feasibility and then remaining elements are implied by sd-envy-freeness and feasibility.

Claim 3: Let $P^3 = (P_1, \hat{P}_2, P_3, \cdots, P_n)$. Then $\varphi(P^3)$ is as follows.

$$\begin{array}{rccccccc} & A_1 & A_2 & A_3 & A_4 & \cdots & A_K & \cdots \\ 1: & \frac{d_1}{n-1} & \frac{d_2 - \frac{d_1+d_2}{n}}{n-1} & \frac{d_3}{n} & \frac{d_4}{n} & \cdots & \frac{d_K}{n} & 0 \\ 2: & 0 & \frac{d_1+d_2}{n} & \frac{d_3}{n} & \frac{d_4}{n} & \cdots & \frac{d_K}{n} & 0 \\ 3\cdots n: & \frac{d_1}{n-1} & \frac{d_2 - \frac{d_1+d_2}{n}}{n-1} & \frac{d_3}{n} & \frac{d_4}{n} & \cdots & \frac{d_K}{n} & 0 \end{array}$$

From P^1 to P^3, agent 2 is the unilateral deviator, and her ranking of A_2 and A_3 are reversed with no other changes. Hence sd-strategy-proofness implies $\varphi_{2A}(P^3) = \varphi_{2A}(P^1)$ for all $A \neq A_1, A_2$. Then sd-envy-freeness implies $\varphi_{iA}(P^3) = \varphi_{2A}(P^3)$ for all $i = 1, 3, \cdots, n$ and all $A \neq A_1, A_2$. In addition, sd-efficiency implies $\varphi_{2A_1}(P^3) = 0$, given which all other elements are implied by sd-envy-freeness and feasibility.

Claim 4: Let $P^4 = (\bar{P}_1, \hat{P}_2, P_3, \cdots, P_n)$. Then $\varphi(P^4)$ is as follows.

$$\begin{array}{rccccccc} & A_1 & A_2 & A_3 & A_4 & \cdots & A_K & \cdots \\ 1: & \frac{d_1}{n-1} & 0 & \frac{d_1+d_2+d_3}{n} - \frac{d_1}{n-1} & \frac{d_4}{n} & \cdots & \frac{d_K}{n} & 0 \\ 2: & 0 & \frac{d_1+d_2+d_3}{n} - \frac{d_3 - \frac{d_2+d_3}{n}}{n-1} & \frac{d_3 - \frac{d_2+d_3}{n}}{n-1} & \frac{d_4}{n} & \cdots & \frac{d_K}{n} & 0 \\ 3\cdots n: & \frac{d_1}{n-1} & \frac{d_1+d_2+d_3}{n} - \frac{d_1}{n-1} - \frac{d_3 - \frac{d_2+d_3}{n}}{n-1} & \frac{d_3 - \frac{d_2+d_3}{n}}{n-1} & \frac{d_4}{n} & \cdots & \frac{d_K}{n} & 0 \end{array}$$

First, from P^3 to P^4, agent 1 is the unilateral deviator, and her ranking of A_2 and A_3 are reversed with no other changes. Hence $\varphi_{1A}(P^4) = \varphi_{1A}(P^3)$ for all $A \neq A_2, A_3$. Second, from P^2 to P^4, agent 2 is the unilateral deviator and she reversed the ranking of A_1 and A_2. Hence $\varphi_{2A}(P^4) = \varphi_{2A}(P^2)$ for all $A \neq A_1, A_2$. Third, sd-envy-freeness implies that $\varphi_{iA}(P^4) = \varphi_{1A}(P^4)$ for all $i = 3, \cdots, n$ and $A \neq A_2, A_3$. In addition sd-envy-freeness implies also that $\varphi_{iA_3}(P^4) = \varphi_{2A_3}(P^4)$ for all $i = 3, \cdots, n$. Fourth, sd-efficiency implies $\varphi_{1A_2}(P^4) =$

$\varphi_{2A_1}(P^4) = 0$. Last, the remaining elements, i.e., $\varphi_{1A_3}(P^4)$ and $\varphi_{iA_2}(P^4)$ for $i = 2, \cdots, n$, are implied by feasibility.

By the fact that there exists $x \in A_1 \cap A_2$ such that $x \notin A_k$ for all $k = 3, \cdots, K$ and that $q_x = d_1 + d_2$, we have the following contradiction.

$$d_1 + d_2 = \sum_{i \in I} \varphi_{iA_1}(P^4) + \varphi_{iA_2}(P^4) \Rightarrow d_1 = 0. \quad \blacksquare$$

Appendix

We noted in the main text that Lemma 1 might be of independent interest. We present here a proof in three steps, each of which is divided into two parts.

Lemma 1. Strong unbalancedness $\underset{\not\Leftarrow}{\Rightarrow}$ sd-efficiency $\underset{\not\Leftarrow}{\Rightarrow}$ weak unbalancedness $\underset{\not\Leftarrow}{\Rightarrow}$ acyclicity.

Step 1.1: Strong unbalancedness $\implies$ sd-efficiency. Let $L \in \mathcal{L}$ be strongly unbalanced at $P \in \mathbb{P}^n$. Suppose L is not sd-efficient at P. Then by Theorem 1, L is balanced at P, which implies the existence of an $\alpha : \mathcal{T} \to \mathbb{R}_+$ such that (i) $\alpha(i, A, B) > 0$ implies $L_{iA} > 0$ and $B \; P_i \; A$, (ii) $\forall x \in X: \quad \sum_{\{(i,A,B) \in \mathcal{T} : x \in B\}} \alpha(i, A, B) = \sum_{\{(i,A,B) \in \mathcal{T} : x \in A\}} \alpha(i, A, B)$.

Let $\mathcal{S} = \{(i, A, B) \in \mathcal{T} : \alpha(i, A, B) > 0\}$. Then by definition, $(i, A, B) \in \mathcal{S}$ implies $L_{iA} > 0$ and $B \; P_i \; A$. We claim that $\forall x \in X$: $d(x, \mathcal{S}) > 0 \Leftrightarrow s(x, \mathcal{S}) > 0$. Suppose not, and let $x \in X$ be such that $d(x, \mathcal{S}) > 0$ and $s(x, \mathcal{S}) = 0$. We identify a contradiction below. (A contradiction can be identified analogously for the other case where $d(x, \mathcal{S}) = 0$ and $s(x, \mathcal{S}) > 0$.)

$$\begin{aligned}
& \sum_{\{(i,A,B) \in \mathcal{T} : x \in B\}} \alpha(i, A, B) - \sum_{\{(i,A,B) \in \mathcal{T} : x \in A\}} \alpha(i, A, B) \\
= & \sum_{\{(i,A,B) \in \mathcal{S} : x \in B\}} \alpha(i, A, B) - \sum_{\{(i,A,B) \in \mathcal{S} : x \in A\}} \alpha(i, A, B) \\
= & \sum_{\{(i,A,B) \in \mathcal{S} : x \in B \setminus A\}} \alpha(i, A, B) - \sum_{\{(i,A,B) \in \mathcal{S} : x \in A \setminus B\}} \alpha(i, A, B) \\
= & \sum_{\{(i,A,B) \in \mathcal{S} : x \in B\}} \alpha(i, A, B) - 0 > 0 : \text{ contradiction.}
\end{aligned}$$

In the above, the first two equations follow from definitions. The third equation follows from $s(x, \mathcal{S}) = 0$ and the last inequality follows from $d(x, \mathcal{S}) > 0$.

Step 1.2: Strong unbalancedness $\not\Longleftarrow$ sd-efficiency. This is shown by the following example. Let $X = \{a, b, c\}$ and $q_x = 1$ for all $x \in X$. Let the preference profile P and the assignment L be as below.

$$\begin{array}{llll@{\qquad}lcccc}
 & & & & & a & b & c & ab \\
P_1: & ab & c & \cdots & L_1: & 0 & 0 & 1 & 0 \\
P_2: & c & b & \cdots & L_2: & 0 & 1 & 0 & 0 \\
P_3: & c & a & \cdots & L_3: & 1 & 0 & 0 & 0
\end{array}$$

We show that L is sd-efficient at P. Suppose not, and let L' dominate L.

First, we show $L'_1 = L_1$. Suppose not, $L'_1 \; P_1^{sd} \; L_1$ implies that $\exists \; \epsilon_1 \in (0, 1]$ s.t.

$$\begin{array}{lcccc} & a & b & c & ab \\ L'_1: & 0 & 0 & 1-\epsilon_1 & \epsilon_1 \end{array}$$

Given this, $L'_2 \; P_2^{sd} \; L_2$ and $L'_3 \; P_3^{sd} \; L_3$ imply the existence of $\epsilon_2, \epsilon_3 \in [0, 1]$ such that

$$\begin{array}{lcccc} & a & b & c & ab \\ L'_2: & 0 & 1-\epsilon_2 & \epsilon_2 & 0 \\ L'_2: & 1-\epsilon_3 & 0 & \epsilon_3 & 0 \end{array}$$

Then feasibility requires $\epsilon_1 + (1 - \epsilon_3) = q_a = 1$ and $\epsilon_1 + (1 - \epsilon_2) = q_b = 1$, which imply $\epsilon_1 = \epsilon_2 = \epsilon_3$. This however implies $1 - \epsilon_1 + \epsilon_2 + \epsilon_3 \neq 1 = q_c$: contradiction.

Given $L'_1 = L_1$, feasibility implies $L'_{2c} = L'_{3c} = 0$ and hence $L'_2 \; P_2^{sd} \; L_2$ and $L'_3 \; P_3^{sd} \; L_3$ imply $L'_2 = L_2$ and $L'_3 = L_3$, which means $L = L'$: contradiction.

Next we show that L is strongly unbalanced at P. To do this, let $\mathcal{S} = \{(1, c, ab), (2, a, c), (3, b, c)\}$. Then it is easy to verify that L is strongly unbalanced at P: (i) $L_{1c} > 0$, $ab \; P_1 \; c$; $L_{2a} > 0$, $c \; P_2 \; a$; $L_{3b} > 0$, $c \; P_3 \; b$; (ii) $d(x, \mathcal{S}) > 0$ and $s(x, \mathcal{S}) > 0$ for all $x \in X$.

Step 2.1: Sd-efficiency $\Longrightarrow$ weak unbalancedness. By Proposition 1, it suffices to show unbalancedness $\Longrightarrow$ weak unbalancedness. Suppose an assignment $L \in \mathcal{L}$ be weakly balanced at $P \in \mathbb{P}^n$. Then there is a subset $\mathcal{S} \subset \mathcal{T}$ such that (i) $(i, A, B) \in \mathcal{S}$ implies $L_{iA} > 0$, $B \; P_i \; A$, and (ii) $\forall x \in X$: $d(x, \mathcal{S}) = s(x, \mathcal{S})$. In the following, we construct a mapping $\alpha : \mathcal{T} \to \mathbb{R}_+$ such that (i) $\alpha(i, A, B) > 0$ implies $L_{iA} > 0$, $B \; P_i \; A$, and (ii) $\forall x \in X$: $\sum_{\{(i,A,B) \in \mathcal{T} : x \in B\}} \alpha(i, A, B) = \sum_{\{(i,A,B) \in \mathcal{T} : x \in A\}} \alpha(i, A, B)$.

In particular, let $\alpha(i, A, B) = \epsilon$ for all $(i, A, B) \in \mathcal{S}$ and $\alpha(i, A, B) = 0$ otherwise, where ϵ is a small positive number. Then (i) is satisfied by definition and (ii) follows from the equation below.

$$\begin{aligned} \forall x \in X : & \sum_{\{(i,A,B) \in \mathcal{T} : x \in B\}} \alpha(i, A, B) - \sum_{\{(i,A,B) \in \mathcal{T} : x \in A\}} \alpha(i, A, B) \\ & = \sum_{\{(i,A,B) \in \mathcal{S} : x \in B\}} \alpha(i, A, B) - \sum_{\{(i,A,B) \in \mathcal{S} : x \in A\}} \alpha(i, A, B) \\ & = \sum_{\{(i,A,B) \in \mathcal{S} : x \in B \setminus A\}} \alpha(i, A, B) - \sum_{\{(i,A,B) \in \mathcal{S} : x \in A \setminus B\}} \alpha(i, A, B) \\ & = \epsilon \cdot d(x, \mathcal{S}) - \epsilon \cdot s(x, \mathcal{S}) = 0. \end{aligned}$$

Step 2.2: Sd-efficiency $\not\Longleftarrow$ weak unbalancedness. This is shown by the following example. Let $X = \{a, b, c, d\}$ and $q_x = 1$ for each $x \in X$. Let the preference profile P and the assignment

L, L' be as below.

$P_1:$	c	$\emptyset$	$\cdots$
$P_2:$	b	a	$\cdots$
$P_3:$	d	b	$\cdots$
$P_4:$	ab	c	$\cdots$
$P_5:$	$\emptyset$	d	$\cdots$

	a	b	c	d	$\emptyset$	ab
$L_1:$	0	0	0	0	1	0
$L_2:$	1	0	0	0	0	0
$L_3:$	0	1	0	0	0	0
$L_4:$	0	0	1	0	0	0
$L_5:$	0	0	0	1	0	0

	a	b	c	d	$\emptyset$	ab
$L'_1:$	0	0	ϵ	0	$1-\epsilon$	0
$L'_2:$	$1-\epsilon$	ϵ	0	0	0	0
$L'_3:$	0	$1-2\epsilon$	0	2ϵ	0	0
$L'_4:$	0	0	$1-\epsilon$	0	0	ϵ
$L'_5:$	0	0	0	$1-2\epsilon$	2ϵ	0

We verify that L is weakly unbalanced at P. Suppose not, and let $\mathcal{S} \subset \mathcal{T}$ be a subset such that (i) $(i, A, B) \in \mathcal{S}$ implies $L_{iA} > 0$ and $B \; P_i \; A$, and (ii) $\forall \; x \in X$, $d(x, \mathcal{S}) = s(x, \mathcal{S})$. Notice that for each $i \in I$, L_i assigns the entire probability to the second ranked bundle according to P_i. Hence $\mathcal{S} \subset \{(1, \emptyset, c), (2, a, b), (3, b, d)), (4, c, ab), (5, d, \emptyset\}$. Next, by definition of weak unbalancedness, if $\mathcal{S}$ is non-empty, it must include all five triples in order to make $d(x, \mathcal{S}) = s(x, \mathcal{S})$ for all $x \in X$. So $\mathcal{S} = \{(1, \emptyset, c), (2, a, b), (3, b, d), (4, c, ab), (5, d, \emptyset)\}$. However, $d(b, \mathcal{S}) = 2 \neq 1 = s(b, \mathcal{S})$: contradiction.

We verify next that L is not sd-efficient at P by showing that L is dominated by L'. Let $\epsilon \in (0, 1]$. Feasibility of L' is evident. Notice that for each $i \in I$, the change from L_i to L'_i is achieved by moving some probability from the second ranked bundle to the top ranked bundle. It follows that L is dominated by L' and hence L is not sd-efficient at P.

Step 3.1: Weak unbalancedness $\implies$ acyclicity. Let $L \in \mathcal{L}$ be weakly unbalanced at $P \in \mathbb{P}^n$ such that $\tau(P, L)$ has a cycle. Let a cycle be as follows

$$A_1 \; \tau(P, L) \; A_2 \; \tau(P, L) \; A_3 \cdots A_{K-1} \; \tau(P, L) \; A_K \; \tau(P, L) \; A_1.$$

In addition, let i_k be such that $A_{k+1} \; P_{i_k} \; A_k$ and $L_{i_k A_k} > 0$. Let $\mathcal{S} = \{(i_k, A_k, A_{k+1}) : k = 1, \cdots, K\}$ with $A_{K+1} = A_1$. Fixing an arbitrary $x \in X$, we prove $d(x, \mathcal{S}) = s(x, \mathcal{S})$. If $x \notin A_k$ for all $k = 1, \cdots, K$, by definition $d(x, \mathcal{S}) = s(x, \mathcal{S}) = 0$. Otherwise, let $(i_{k-1}, A_{k-1}, A_k) \in \mathcal{S}$ be arbitrary such that $x \in A_k \backslash A_{k-1}$. It suffices to show the existence of $(i_{k+l}, A_{k+l}, A_{k+l+1}) \in \mathcal{S}$ such that $x \in A_{k+l} \backslash A_{k+l+1}$. By the fact that $A_1, \cdots, A_K$ forms a cycle, such a triple exists.

Step 3.2: Weak unbalancedness $\not\Leftarrow$ acyclicity. This is shown by the following example. Let $A = \{a, b, c\}$, $q = (1, 1, 1)$, $I = \{1, 2\}$. Let the preferences of two agents be

$$\begin{array}{lcccccccc} P_1: & c & a & ab & b & \emptyset & bc & ac & abc \\ P_2: & a & c & ab & b & \emptyset & bc & ac & abc \end{array}$$

Consider a random assignment L below.

	c	a	ab	b	$\emptyset$	bc	ac	abc
$L_1:$	0	0	0.2	0	0.3	0	0	0.5
$L_2:$	0.2	0	0	0	0.5	0	0	0.3

We have stated in Example 8 that L is acyclic at P. Let $\mathcal{S} = \{(1, ab, c), (1, \emptyset, b), (2, c, a)\}$. We have $c \; P_1 \; ab$, $L_{1ab} > 0$, $b \; P_1 \; \emptyset$, $L_{1\emptyset} > 0$, $a \; P_2 \; c$, and $L_{2c} > 0$. In addition, it follows by counting that $d(x, \mathcal{S}) = s(x, \mathcal{S}) = 1$ for all $x \in \{a, b, c\}$. Hence L is not weakly balanced at P.

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