Local vs. Global Strategy-Proofness: A New Equivalence Result for Ordinal Mechanisms*

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Abstract

We study two incentive compatibility notions of ordinal mechanisms: strategy-proofness and a new notion of local strategy-proofness, called block-adjacent strategy-proofness, which requires no gain by a flip of two adjacent blocks. A condition on the preference domain called *path-nestedness* is identified as sufficient for the equivalence between these two notions.

Keywords: ordinal mechanism; block-adjacent strategy-proofness; path-nested;

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1 Introduction

Ordinal mechanisms collect agents' ordinal preferences on a finite set of alternatives and specify an outcome. Examples include the voting mechanisms (Gibbard, 1973; Satterthwaite, 1975; Gibbard, 1977), the assignment mechanisms (Pápai, 2000; Bogomolnaia and Moulin, 2001), etc. A central property we want an ordinal mechanism to satisfy is that no one gains by misrepresenting her true preference, called strategy-proofness. This property is in general difficult to verify since its verification requires comparisons between the outcomes induced by *every* pair of preferences, given an arbitrary profile of other agents' preferences. To simplify the task, a notion of local strategy-proofness was studied, requiring that no one gains by a flip of two adjacently ranked alternatives. An equivalence between these two notions is surely interesting, as it dramatically simplifies the verification of strategy-proofness. Apparently, whether such an equivalence holds depends on the preference domain on which the mechanisms are defined. Sato (2013) focused on deterministic mechanisms and proved that, if a domain satisfies

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connectedness and non-restoration, the equivalence holds. Cho (2016) found that these conditions remain sufficient for random mechanisms.[1] A domain is connected if, from an arbitrary preference to another, there is a sequence of preferences in the domain such that every pair of contiguous preferences are different only in a flip between two adjacently ranked alternatives. Moreover, a connected domain satisfies non-restoration if no pair of alternatives are flipped more than once along this sequence. These equivalence results, however, cannot be invoked when connectedness or non-restoration is violated. In particular, many interesting domains are not connected. In response to this fact, we introduce a new local strategy-proofness and prove a new equivalence, which is presumably useful when either one of the two conditions is violated. Liu (2019) studied the random assignment problems and proved that, for a particular class of domains, the mechanism introduced by Bogomolnaia and Moulin (2001) satisfies the local strategy-proofness defined in the current note, which hence implies global strategy-proofness. Therefore, Liu (2019) can be seen as an application of the equivalence in this note.

2 Preliminaries

Let A be a finite set of alternatives with $|A| = m \geqslant 2$.[2] Let $\mathbb{P}$ be the collection of all linear orders on A, i.e., complete, transitive, and antisymmetric binary relations on A. We call these linear orders preferences and $\mathbb{P}$ the universal domain. Besides, a preference such that $a\ P\ b$ and $b\ P\ c$ is usually denoted abc. For a specific mechanism design problem, the set of admissible preferences is called the **domain** and denoted $\mathbb{D} \subset \mathbb{P}$. Given $P \in \mathbb{P}$, $r_k(P)$ denotes the k-th ranked alternative according to P.

An ordinal mechanism is a mapping $\varphi : \mathbb{D} \to \Delta(A)$, where $\Delta(A)$ denotes the set of lotteries on A. The preferences on alternatives are extended to preferences on lotteries by stochastic dominance. In particular, given $P \in \mathbb{D}$ and $\lambda, \lambda' \in \Delta(A)$, λ stochastically dominates λ' according to P, denoted $\lambda\ \ P^{sd}\ \ \lambda'$, if $\sum_{l=1}^{k} \lambda_{r_l(P)} \geqslant \sum_{l=1}^{k} \lambda'_{r_l(P)}$ for all $1 \leqslant k \leqslant n$.[3] An ordinal mechanism $\varphi : \mathbb{D} \to \Delta(A)$ is **sd-strategy-proof (sd-SP)** if for all $P, P' \in \mathbb{D}$, $\varphi(P)\ \ P^{sd}\ \ \varphi(P')$. An ordinal mechanism is deterministic if it selects only degenerate lotteries and in this case, sd-strategy-proofness is called strategy-proofness (SP).[4]

[1]Carroll (2012) also studied the equivalence between local and global strategy-proofness. However, Carroll (2012) studied specific economic settings and investigated not only ordinal mechanisms but also cardinal mechanisms. Recently, Kumar et al. (2019) studied deterministic voting mechanisms, introduced a general notion of local strategy-proofness, and provided a condition on preference domains called "lower contour set no-restoration property", which is shown sufficient and necessary for the equivalence between strategy-proofness and their local strategy-proofness. However, this equivalence is invalid for random mechanisms, as noted by Theorem 2 in their paper.

[2]If $|A| = 1$, the mechanism is constant and hence strategy-proofness is trivially satisfied.

[3]Equivalently, $\lambda\ P^{sd}\ \lambda'$ iff, $\forall\ u$ representing P, $E_u(\lambda) \geqslant E_u(\lambda')$, where $E_u(\lambda)$ denotes the expected utility delivered by λ given u.

[4]Mechanism design problems usually involve multiple agents. However, the properties we study in this note concern only individual agent's incentive when she decides which preference to report, given other agents' reported preferences. Therefore, for simplicity, the mechanism defined here collects only one agent's preference and specifies one outcome for this agent. However, our results straightforwardly extend to multi-agent problems.

3 Results

We introduce first the local strategy-proofness we study. We say a subset of alternatives clusters in a preference if they are adjacently ranked. Formally, $B \subset A$ clusters in $P \in \mathbb{D}$ if $\forall a, b \in B, \nexists\, x \in A \backslash B$ such that $a\ P\ x\ P\ b$ or $b\ P\ x\ P\ a$. We call these subsets blocks.

Definition 1. *Two preferences $P, P' \in \mathbb{D}$ are **block-adjacent** if there are two nonempty and disjoint subsets $A_1, A_2 \subset A$ such that*

1. *A_1, A_2, and $A_1 \cup A_2$ cluster in P and P'.*
2. *$\forall a, b \in A$ such that $a \in A_1$ and $b \in A_2$, $a\ P'\ b \Leftrightarrow b\ P\ a$.*
3. *$\forall a, b \in A$ such that either $a \notin A_1$ or $b \notin A_2$, $a\ P'\ b \Leftrightarrow a\ P\ b$.*

Note that two preferences are either not block-adjacent or block-adjacent with respect to a unique pair of blocks. Note also that the adjacencies used by Sato (2013) and Cho (2016) are special cases of block-adjacencies where the flipped blocks are singletons. We denote the blocks flipped between P and P' as $FB_1(P, P')$ and $FB_2(P, P')$, where FB refers to "flipped blocks." An ordinal mechanism $\varphi : \mathbb{D} \to \Delta(A)$ is **block-adjacent sd-strategy-proof (BA-sd-SP)** if for all block-adjacent $P, P' \in \mathbb{D}$, $\varphi(P)\ P^{sd}\ \ \varphi(P')$. When the mechanism is deterministic, this notion is called block-adjacent strategy-proof (BA-SP). The local strategy-proofness studied by Sato (2013) and Cho (2016) are called respectively "AM strategy-proofness" and "sd-adjacent strategy-proofness", where "AM" stands for adjacent manipulation. Since block-adjacency is a generalization of the adjacency notion they use, BA-SP is stronger than the AM strategy-proofness of Sato (2013) and BA-sd-SP is stronger than the sd-adjacent strategy-proofness of Cho (2016).

We proceed to introduce two conditions on preference domains. First, a preference domain $\mathbb{D}$ is **block-connected** if for any two preferences $P, P' \in \mathbb{D}$, there exist $P_1, \cdots, P_M \in \mathbb{D}$ such that (i) $P_1 = P$, (ii) $P_M = P'$, and (iii) $\forall\ m = 1, \cdots, M-1$, P_m and P_{m+1} are block-adjacent. We call this sequence a **path** from P to P'. To introduce the second condition, let A_1, A_2 and A_3, A_4 be two pairs of nonempty and disjoint subsets, i.e., $\emptyset \neq A_1, A_2, A_3, A_4 \subset A$, $A_1 \cap A_2 = \emptyset$, and $A_3 \cap A_4 = \emptyset$. We say A_1, A_2 are nested in or disjoint from A_3, A_4, denoted $\{A_1, A_2\} \sqsubseteq \{A_3, A_4\}$, if either $A_1 \cup A_2 \subset A_3$, or $A_1 \cup A_2 \subset A_4$, or $(A_1 \cup A_2) \cap (A_3 \cup A_4) = \emptyset$. A path $P_1, \cdots, P_M$ is nested if the blocks flipped later are nested in or disjoint from the blocks flipped earlier. Formally, $\forall\ 1 \leqslant m' < m \leqslant M-1$, $\{FB_1(P_m, P_{m+1}), FB_2(P_m, P_{m+1})\} \sqsubseteq \{FB_1(P_{m'}, P_{m'+1}), FB_2(P_{m'}, P_{m'+1})\}$. Finally, A domain $\mathbb{D}$ is **path-nested** if, for all distinct $P, P' \in \mathbb{D}$, there is a nested path from P to P'. An example is in Figure 1.

We are now ready to present our main result below.

Theorem 1. *BA-sd-SP is equivalent to sd-SP on path-nested domains.*

We close this section by highlighting three points. First, for deterministic mechanisms, a corollary of Theorem 1 is that BA-SP is equivalent to SP on path-nested domains. Second, block-connectedness is necessary for the equivalence, while path-nestedness is not. Third, there is no logical relation between path-nestedness and non-restoration of Sato (2013). Details can be found in the working paper version (Liu, 2017).

4 Examples and Discussions

4.1 A Path-Nested but not Connected Domain

Figure 1 below illustrates a sequentially dichotomous domain (Liu, 2019), where the bold-faced sequences are preferences and the dotted lines indicate block-adjacencies. A set of preferences is a sequentially dichotomous domain, if there is a fixed sequence of attributes satisfying the following requirements. First, the set of alternatives can be partitioned into two subsets, one containing the alternatives possessing the first attribute and the other containing the alternatives not. It is hence required that, for every preference in the domain, the alternatives in one subset are all better than the ones in the other subset. Second, within each subset in the first step, the alternatives can be partitioned further into two smaller subsets according to the second attribute and it is still required that the alternatives in one subset are all better than the ones in the other. It proceeds until no partition can be made. One can easily verify that the domain in Figure 1 is not connected. Hence, the equivalence results by Sato (2013) and Cho (2016) cannot be invoked. However, such a domain is path-nested and hence Theorem 1 applies.

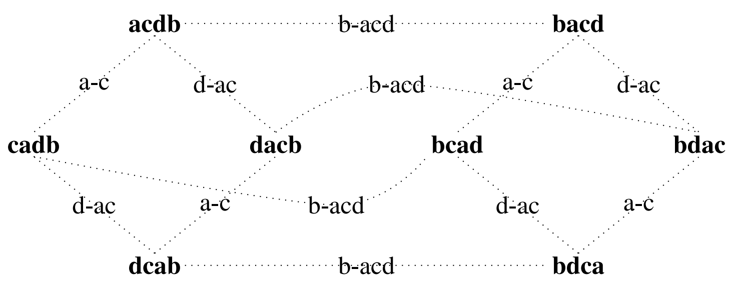

Figure 1: A Path-Nested Domain.

4.2 A Path-Nested and Connected Domain with Restoration

The domain in Figure 2 has been discussed by Sato (2013). The solid lines illustrate adjacencies, indicating that this domain satisfies connectedness but violates non-restoration. Block-adjacency allows us to draw two more links, indicating path-nestedness. Theorem 1 then applies.

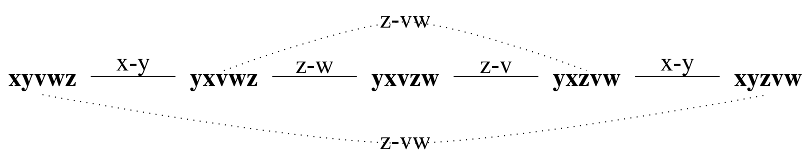

Figure 2: A Connected Domain that Violates non-Restoration

4.3 Regular Domains

We start with an observation below. Let $B \subset A$ be an arbitrary subset of alternatives with $|B| \geqslant 2$. A dichotomous partition of B refers to a pair of subsets $\emptyset \neq B_1, B_2 \subset B$ such that $B_1 \cup B_2 = B$ and $B_1 \cap B_2 = \emptyset$.

Proposition 1. *Let $B \subset A$ be such that $|B| \geqslant 2$ and $P, P' \in \mathbb{P}$ distinct. Then, exactly one of the following three happens.*

1. *$\exists$ a dichotomous partition $B_1, B_2 \subset B$ s.t. $\forall\, b_1 \in B_1$ and $b_2 \in B_2$, $b_1\ P\ b_2$ and $b_1\ P'\ b_2$.*
2. *$\exists$ a dichotomous partition $B_1, B_2 \subset B$ s.t. $\forall\, b_1 \in B_1$ and $b_2 \in B_2$, $b_1\ P\ b_2$ and $b_2\ P'\ b_1$.*
3. *$\exists$ distinct $a, b, c, d \in B$ s.t. $a\ P\ b\ P\ c\ P\ d$ and $[b\ P'\ d\ P'\ a\ P'\ c$ or $c\ P'\ a\ P'\ d\ P'\ b]$.*

Recall that the nested paths involve essentially block flips, meaning that the third case above potentially prevents a domain to be path-nested. Hence, we say a domain $\mathbb{D}$ is **irregular** if case 3 happens, i.e., there exist four alternatives and two preferences showing the pattern in the third bullet. A domain is **regular** if it is not irregular. Then we have the following.

Proposition 2. *Any path-nested domain is regular.*

The converse of the above proposition is not true since *path-nestedness* requires some sort of richness, which is not provided by regularity. For instance, if $bcad$ is removed from the domain in Figure 1, regularity is satisfied but path-nestedness is violated. However, a domain is path-nested if it is maximally regular, in the sense that it becomes irregular whenever an additional preference is added.

Proposition 3. *Any maximally regular domain is path-nested.*

5 Proofs

5.1 Proof of Theorem 1

The necessity part is evident by definition. We prove the sufficiency. Let $\mathbb{D}$ be a *path-nested* domain and $\varphi : \mathbb{D} \to \Delta(A)$ a BA-sd-SP mechanism. What we need to show is, $\forall P, P' \in \mathbb{D}$, $\varphi(P)\ P^{sd}\ \varphi(P')$. Fixing an arbitrary pair $P, P' \in \mathbb{D}$, let $P = P_1, \cdots, P_M = P'$ be an arbitrary nested path. Then, the transitivity of P_1^{sd} implies that it suffices to show, $\forall\, m = 1, \cdots, M-1$, $\varphi(P_m)\ P_1^{sd}\ \varphi(P_{m+1})$. Note that $\varphi(P_1)\ P_1^{sd}\ \varphi(P_2)$ is implied directly by BA-sd-SP. Fixing $m = 2, \cdots, M-1$, $\varphi(P_m)\ P_1^{sd}\ \varphi(P_{m+1})$ is implied by the following induction.

Initial Statement: $\varphi(P_m)\ P_m^{sd}\ \varphi(P_{m+1})$ by BA-sd-SP.

Induction Statement: $\forall 2 \leqslant \alpha \leqslant m$, $\varphi(P_m)\ P_\alpha^{sd}\ \varphi(P_{m+1}) \Rightarrow \varphi(P_m)\ P_{\alpha-1}^{sd}\ \varphi(P_{m+1})$.

To simplify notations, we define the following.

$A_1 \equiv FB_1(P_{\alpha-1}, P_\alpha) \quad B_1 \equiv FB_1(P_\alpha, P_{\alpha+1}) \quad C_1 \equiv FB_1(P_m, P_{m+1})$

$A_2 \equiv FB_2(P_{\alpha-1}, P_\alpha) \quad B_2 \equiv FB_2(P_\alpha, P_{\alpha+1}) \quad C_2 \equiv FB_2(P_m, P_{m+1})$

To prove the induction statement, we need to consider three cases: (1) $B_1 \cup B_2 \subset A_1$, (2) $B_1 \cup B_2 \subset A_2$, and (3) $[B_1 \cup B_2] \cap [A_1 \cup A_2] = \emptyset$. We prove the statement for case 1 illustrated below, where X and Y denote the common upper and lower contour sets. The same logic applies to other cases.

$$P_{\alpha-1}: \quad X \succ \underbrace{\cdots \succ B_1 \succ B_2 \succ \cdots}_{A_1} \succ A_2 \succ Y$$
$$P_{\alpha}: \quad X \succ A_2 \succ \underbrace{\cdots \succ B_1 \succ B_2 \succ \cdots}_{A_1} \succ Y$$

Given $\varphi(P_m)\ P_m^{sd}\ \varphi(P_{m+1})$ and $\varphi(P_{m+1})\ P_{m+1}^{sd}\ \varphi(P_m)$, $\forall\ x \notin C_1 \cup C_2$, $\varphi_x(P_m) = \varphi_x(P_{m+1})$. Put otherwise, $\varphi(P_m)$ and $\varphi(P_{m+1})$ can be different only on the probabilities of alternatives in $C_1 \cup C_2$. Note also that the path-nestedness implies that $C_1 \cup C_2$ clusters in both $P_{\alpha-1}$ and P_α. Moreover, $P_{\alpha-1}$ and P_α differ only in a flip between A_1 and A_2. In particular, the ranking of the alternatives in $C_1 \cup C_2$ is the same. Given these, one can verify by the definition of stochastic dominance that $\varphi(P_m)\ P_\alpha^{sd}\ \varphi(P_{m+1})$ implies $\varphi(P_m)\ P_{\alpha-1}^{sd}\ \varphi(P_{m+1})$.

5.2 Proof of Proposition 1

If $|B| = 2$, it is trivial that either case 1 or 2 happens. If $|B| = 3$, without loss of generality, let P be abc. Then P' can be one of the following five: acb, bac, bca, cab, and cba. It is easy to check that either case 1 or 2 happens. If $|B| \geqslant 4$, we prove that the negation of case 3 implies either case 1 or 2. Pick arbitrary $a, b, c, d \in B$ such that $a\ P\ b\ P\ c\ P\ d$. Then the negation of case 3 implies neither $b\ P'\ d\ P'\ a\ P'\ c$ nor $c\ P'\ a\ P'\ d\ P'\ b$. Given this, one can verify that, for each possible ranking of these alternatives in P', either case 1 or 2 happens. For example, if $b\ P'\ c\ P'\ a\ P'\ d$, case 2 happens with $B_1 = \{a, b\}$ and $B_2 = \{c, d\}$.

5.3 Proof of Proposition 2

We prove the contrapositive statement: Any irregular domain is not path-nested. Let $P, P' \in \mathbb{D}$ be such that $a\ P\ b\ P\ c\ P\ d$ and $b\ P'\ d\ P'\ a\ P'\ c$. The other case is similar. Suppose $\mathbb{D}$ is path-nested and let $P = P_1, P_2, \cdots, P_M = P'$ be a nested path. Let the ranking of a, b, c, d be changed for the first time from P_m to P_{m+1}. To simplify notation, let $A_1 = FB_1(P_m, P_{m+1})$ and $A_2 = FB_2(P_m, P_{m+1})$. First, we claim $\{a, b, c, d\} \subset A_1 \cup A_2$. To see this, suppose $A_1 \cup A_2$ contains only a proper subset of $\{a, b, c, d\}$, say $d \notin A_1 \cup A_2$. Then, by definition of path-nestedness, the ranking between a and d will not be reversed along the path and hence $a\ P'\ d$: contradiction. Second, given $\{a, b, c, d\} \subset A_1 \cup A_2$, we have $a \in A_1$ and $d \in A_2$. Then, we need to consider three cases: (i) $b \in A_1$ and $c \in A_2$, (ii) $b, c \in A_1$, and (iii) $b, c, \in A_2$. For case (i), by definition of path-nestedness, $c\ P_{m+1}\ b$ and the ranking between c and b will not be reversed along the path. Hence $c\ P'\ b$: contradiction. Similar contradictions can be found for the other two cases, which complete the proof.

5.4 Proof of Proposition 3

Let $\mathbb{D} \subset \mathbb{P}$ denote a maximally regular domain. Pick arbitrary $P, P' \in \mathbb{D}$, we show the existence of a nested path $P = P_1, P_2, \cdots, P_M = P' \in \mathbb{D}$. We show the identification of P_2 and, $\forall\ m = 3, \cdots, M$, P_m can be identified by replacing the role of P_1 below with P_{m-1}.

Let $A_1, \cdots, A_K$ be the finest partition of A such that all these blocks cluster in both P_1 and P_M and they are ranked in the same way. Such a partition uniquely exists. Let A_k be the first block such that $|A_k| \geqslant 2$. Then, treating A_k as B in Proposition 1, then the regularity of $\mathbb{D}$ implies that either case 1 or case 2 happens. Moreover, since the partition is the finest, case 1 is impossible. Hence let $\{B_1, B_2\}$ be the corresponding partition of A_k and let $B_1\ P_1\ B_2$. Then, let P_2 be the same as P_1 except for a flip between B_1 and B_2. What remains is to prove $P_2 \in \mathbb{D}$. Suppose not. Then the fact that $\mathbb{D}$ is maximally regular implies the existence of $\tilde{P} \in \mathbb{D}$ such that P_2 and $\tilde{P}$ form a situation illustrated by case 3 in Proposition 1. However, if this is true, is it easy to see that P_1 and $\tilde{P}$ also form such a situation, which contradicts the regularity of $\mathbb{D}$.

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