Random Assignments on Preference Domains with a Tier Structure∗

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Abstract

We address a standard random assignment problem (Bogomolnaia and Moulin, 2001). A weakly connected domain admitting an sd-strategy-proof, sd-efficient and equal-treatment-of-equals rule is characterized to be a restricted tier domain. Conversely, on such a domain, the probabilistic serial rule is uniquely characterized by either sd-strategy-proofness, sd-efficiency and equal treatment of equals, or sd-efficiency and sd-envy-freeness. Moreover, we provide an algorithm to construct unions of multiple restricted tier domains, each of which admits an sd-strategy-proof, sd-efficient and sd-envy-free rule.

Keywords: sd-strategy-proofness; sd-efficiency; equal treatment of equals; sd-envy-freeness; probabilistic serial rule; restricted tier domains

JEL Classification: C78, D71.

1 Introduction

We consider the problem of allocating several indivisible objects to a group of agents, each of whom receives at most one object. Each agent reports a strict ordinal preference on objects to the planner, and then the planner assigns a lottery over objects to each agent. The profile of lotteries agents receive is called a random assignment. To extend a preference on objects to an assessment on lotteries, the stochastic dominance extension introduced by Gibbard (1977) is widely

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†Classical examples include assigning college seats to applicants (Gale and Shapley, 1962), houses to residents (Shapley and Scarf, 1974), and jobs to workers (Hylland and Zeckhauser, 1979). Also see papers of Svensson (1999), Pápai (2000), Ehlers (2002) and Pycia and Unver (2017).
adopted: A lottery is viewed at least as good as another one if the former (first-order) stochastically dominates the latter according to the ordinal preference over objects. Equivalently, under the von-Neumann-Morgenstern hypothesis, a lottery stochastically dominates another one if and only if it delivers an expected utility weakly higher than that delivered by the opponent for every cardinal utility representing the ordinal preference.

With the stochastic dominance extension, several axioms are defined for designing random assignment rules which associate each profile of reported preferences to a random assignment. First, \textit{sd-efficiency} requires that no reallocation can be arranged such that all agents are at least as well as before, and someone receives a strictly better lottery. Second, random assignment rules should provide incentives for agents to truthfully reveal their preferences. Accordingly, \textit{sd-strategy-proofness} is introduced, saying that for each agent, the lottery delivered by truth-telling stochastically dominates the lottery induced by any preference misrepresentation, regardless of others’ preferences. In addition, ex ante fairness in the sense of either \textit{equal treatment of equals} or \textit{sd-envy-freeness} is imposed. As suggested by the names, \textit{equal treatment of equals} requires that agents reporting the same preferences receive the same lottery, while \textit{sd-envy-freeness} is stronger, and requires that an agent always weakly prefers her own lottery to others’.

Two classic random assignment rules have been widely studied in the literature: the \textit{random serial dictatorship rule} (Abdulkadiroğlu and Sönmez, 1998) and the \textit{probabilistic serial rule} (Crès and Moulin, 2001; Bogomolnaia and Moulin, 2001). On the one hand, the random serial dictatorship rule is \textit{sd-strategy-proof} and \textit{equal-treatment-of-equals}, but not \textit{sd-efficient} (see Abdulkadiroğlu and Sönmez, 2003; Kesten, 2009). On the other hand, the probabilistic serial rule is \textit{sd-efficient} and \textit{sd-envy-free}, but fails \textit{sd-strategy-proofness}. Moreover, an impossibility result has been established by Bogomolnaia and Moulin (2001): When the numbers of objects and agents are identical and at least four, and agents’ preferences are from the \textit{universal domain} with no restriction, no random assignment rule satisfies \textit{sd-strategy-proofness}, \textit{sd-efficiency} and \textit{equal treatment of equals}. Recently, this impossibility has also been established on some restricted preference domains, e.g., single-peaked domains and single-dipped domains (see Kasajima, 2013; Altuntaş, 2016; Chang and Chun, 2017).

These impossibilities raise a natural question: Is there a reasonable restricted preference domain which admits an \textit{sd-strategy-proof}, \textit{sd-efficient} and \textit{equal-treatment-of-equals} random assignment rule? Furthermore, if the answer is in the affirmative, what are the admissible random assignment rules? This paper provides answers to these two questions.

We execute our investigation in a class of rich domains, \textit{weakly connected} domains, which occupies a prominent position in the literature (see Remark 1). Two preferences are called \textit{neighbors} if across these two preferences, several pair(s) of contiguously ranked objects are locally switched, and all other objects are identically ranked. A domain is then said \textit{weakly connected} if any two distinct preferences are connected via a sequence of preferences in the domain which are consecutively neighbored. This implies that the difference of any two preferences in a \textit{weakly connected} domain can be reconciled via a sequence of local switchings. Our first main result (Theorem 1) shows that a \textit{weakly connected} domain admitting an \textit{sd-strategy-proof}, \textit{sd-efficient}, and \textit{equal-treatment-of-equals} rule must be a \textit{restricted tier domain}. To construct a \textit{restricted tier domain}, objects are first partitioned into several blocks, each of which contains one or two objects. Then, all preferences are required to respect a common ranking of blocks, referred to as a \textit{restricted tier domain}.\footnote{Other preference extensions are also investigated (e.g., Aziz et al., 2014; Bogomolnaia, 2015; Cho, 2018).}
structure. As an example, consider a skyscraper with two apartments on each floor. A restricted tier structure can be elicited according to floors from the top down to the bottom: All agents prefer higher apartments to lower ones. Between two apartments on the same floor, the preferences are arbitrary across agents. The second main result (Theorem 2) searches for the desirable rules on a restricted tier domain, and characterizes the probabilistic serial rule by either sd-strategy-proofness, sd-efficiency and equal treatment of equals, or sd-efficiency and sd-envy-freeness.

Logically, our domain characterization result identifies, within the class of weakly connected domains, the exact boundary between the ones admitting desirable rules and the ones not. Normatively, we treat it as a negative result since a restricted tier domain is so restrictive that agents are almost required to have the same preference. However, we believe that our domain characterization result is critically different from and more informative than all existing impossibility results alluded above. First, it implies every existing impossibility result. One can see this by simply verifying weak connectedness and the failure of the restricted tier structure. Second, our domain characterization result implies the nonexistence of sd-strategy-proof, sd-efficient and equal-treatment-of-equals rules on some important domains that have not been studied by the literature of random assignment (see Remark 1). Third, our domain characterization is potentially useful in distinguishing possibility and impossibility when one in the future studies a specific interesting assignment problem, and encounters with a particular restricted preference domain. Last, it suggests that to find a reasonable restricted domain which admits a desirable rule, we have to go beyond the weakly connected domains. More specifically, given an arbitrary domain (not necessarily weakly connected), we partition it into multiple weakly connected subdomains which are mutually disconnected. Then, the existence of an sd-strategy-proof, sd-efficient and equal-treatment-of-equals rule implies that each subdomain must be a restricted tier domain. In other words, any domain admitting an sd-strategy-proof, sd-efficient and equal-treatment-of-equals rule must be a union of restricted tier domains. Following this direction, we provide an algorithm which gradually excludes preferences from the universal domain, and eventually generates a union of restricted tier domains which is not weakly connected. More importantly, we show that every domain generated by the algorithm is equivalent to a sequentially dichotomous domain of Liu (2018), and hence admits an sd-strategy-proof, sd-efficient and sd-envy-free rule (see Proposition 1).

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 presents the two main results. Section 4 studies the union of restricted tier domains, while Section 5 concludes. The Appendix gathers the omitted proofs.

2 Model

Let \( A \equiv \{a, b, c, d, \ldots \} \) be a finite set of objects and \( I \equiv \{1, 2, \ldots, n\} \) be a finite set of agents. We assume \( |A| = |I| = n \geq 4 \). Each agent \( i \) is equipped with a (strict) preference \( P_i \) over \( A \) which is complete, transitive and antisymmetric, i.e., a linear order. Given \( a, b \in A \), \( a P_i b \) is interpreted as \( "a \) is strictly preferred to \( b \) according to \( P_i" \), and \( a P_i! b \) denotes that \( "a \) is strictly

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3 Starting from an arbitrary preference in the given domain, we repeatedly include in a subdomain preferences which are neighbors to any preference already included. When no more preference can be included, we identify a weakly connected subdomain. The same inclusion process can be repeatedly applied to the remaining preferences, if any, to identify all weakly connected subdomains. In this manner, we partition the given domain into multiple subdomains such that every subdomain is weakly connected, and no one preference in any subdomain is a neighbor to any preference in another subdomain.
preferred to \( b \) according to \( P_i \), and \( a \) and \( b \) are contiguously ranked in \( P_i'' \), i.e., \( a P_i b \), and there exists no \( c \in A \) such that \( a P_i c \) and \( c P_i b \). Let \( r_k(P_i) \), \( k = 1, \ldots, n \), denote the \( k \)-th ranked object in preference \( P_i \). Let \( B(P_i, a) = \{ x \in A : x P_i a \} \) denote the (strict) upper contour set of \( a \) at \( P_i \). Let \( \mathbb{P} \) denote the set of all preferences. The set of admissible preferences is \( \mathbb{D} \subseteq \mathbb{P} \), referred to as a preference domain. In particular, \( \mathbb{P} \) is called the universal domain, and a proper subset of \( \mathbb{P} \) is called a restricted domain. We assume that all agents have the same preference domain \( \mathbb{D} \). A preference profile \( P \equiv (P_1, \ldots, P_n) \equiv (P_i, P_{-i}) \in \mathbb{D}^n \) is an \( n \)-tuple of admissible preferences.

Let \( \Delta(A) \) denote the set of lotteries over \( A \). Given \( \lambda \in \Delta(A) \), \( \lambda_a \) denotes the probability allotted to the object \( a \). A (random) assignment is a bi-stochastic matrix \( L \equiv [L_{ia}]_{i \in I, a \in A} \), namely a non-negative square matrix whose elements in each row and each column sum to unity respectively, i.e., (i) \( L_{ia} \geq 0 \) for all \( i \in I \) and \( a \in A \), (ii) \( \sum_{a \in A} L_{ia} = 1 \) for all \( i \in I \), and (iii) \( \sum_{i \in I} L_{ia} = 1 \) for all \( a \in A \). An element \( L_{ia} \) is interpreted as the probability of agent \( i \) receiving object \( a \). Then, the \( i \)-th row of \( L \), denoted \( L_i \), specifies agent \( i \)'s lottery over \( A \). Let \( \mathcal{L} \) denote the set of all bi-stochastic matrices. The Birkhoff-von-Neumann theorem states that every bi-stochastic matrix can be decomposed as a lottery over permutation matrices. Hence, a random assignment of all bi-stochastic matrices. Given \( I \) to as a preference domain. Let \( \mathcal{B} \) denote the set of all bi-stochastic matrices. The Birkhoff-von-Neumann theorem states that every bi-stochastic matrix can be decomposed as a lottery over permutation matrices. Hence, a random assignment can be implemented by randomly drawing a permutation matrix from a Birkhoff-von-Neumann decomposition and then allocating deterministic objects accordingly. A (random assignment) rule is a mapping \( \varphi : \mathbb{D}^n \to \mathcal{L} \) which specifies a random assignment at each profile of reported preferences. Given \( P \in \mathbb{D}^n \), \( \varphi_{ia}(P) \) denotes the probability of agent \( i \) receiving object \( a \), and \( \varphi_i(P) \) denotes the lottery assigned to agent \( i \). For notational convenience, given a subset \( B \subseteq A \), let \( \varphi_{iB}(P) \equiv \sum_{a \in B} \varphi_{ia}(P) \) denote the probability of agent \( i \) receiving an object in \( B \) at profile \( P \).

Agents assess lotteries according to (first-order) stochastic dominance. Formally, given \( P_i \in \mathbb{D} \) and lotteries \( \lambda, \lambda' \in \Delta(A) \), \( \lambda \) stochastically dominates \( \lambda' \) according to \( P_i \), denoted \( \lambda P_i^{sd} \lambda' \), if \( \sum_{i=1}^k \lambda_{ri}(P_i) \geq \sum_{i=1}^k \lambda'_r(P_i) \) for all \( k = 1, \ldots, n \). Given \( P \in \mathbb{D}^n \), an assignment \( L \) is sd-efficient if there exists no other \( L' \in \mathcal{L} \) Pareto dominating \( L \), i.e., \( L' \neq L \) and \( L_i' P_i^{sd} L_i \) for all \( i \in I \). Accordingly, a rule \( \varphi : \mathbb{D}^n \to \mathcal{L} \) is sd-efficient (or sd-Eff) if \( \varphi(P) \) is sd-efficient for all \( P \in \mathbb{D}^n \). Next, a rule is sd-strategy-proof if for every agent, her lottery under truth-telling always stochastically dominates her lottery induced by any misrepresentation according to her true preference, regardless of others’ preference reporting. Formally, a rule \( \varphi : \mathbb{D}^n \to \mathcal{L} \) is sd-strategy-proof (or sd-SP) if for all \( i \in I \), \( P_i, P'_i \in \mathbb{D} \) and \( P_{-i} \in \mathbb{D}^{n-1} \), \( \varphi_i(P_i, P_{-i}) P_i^{sd} \varphi_i(P'_i, P_{-i}) \). Last, for fairness, given \( P \in \mathbb{D}^n \), an assignment \( L \) is said equal-treatment-of-equals if all agents reporting the same preference receive the same lottery, i.e., for all \( i, j \in I \), \( [P_i = P_j] \Rightarrow [L_i = L_j] \), and a rule \( \varphi : \mathbb{D}^n \to \mathcal{L} \) is equal-treatment-of-equals (or ETE) if \( \varphi(P) \) is equal-treatment-of-equals for all \( P \in \mathbb{D}^n \). As a stronger notion of fairness, an assignment \( L \) is sd-envy-free if every agent treats her own lottery weakly better than any other’s, i.e., \( L_i P_i^{sd} L_j \) for all \( i, j \in I \). Accordingly, a rule \( \varphi : \mathbb{D}^n \to \mathcal{L} \) is sd-envy-free (or sd-EF) if \( \varphi(P) \) is sd-envy-free for all \( P \in \mathbb{D}^n \). Henceforth, we adopt the mentioned abbreviations of these four axioms throughout the paper.

### 2.1 The Probabilistic Serial Rule

In this section, we formally introduce an important random assignment rule, the probabilistic serial (or PS) rule, which selects a random assignment at a given preference profile by the simultaneous eating algorithm with the uniform speed. Such an algorithm hypothetically treats the objects infinitely divisible and specifies the random assignment by an iterative procedure. Starting from time 0, every agent consumes her favorite object at the uniform speed, until an object reaches its
exhaustion. Then, every agent resumes consuming her favorite object in the remaining ones at the uniform speed until another object is exhausted. This procedure is repeated until all objects are exhausted. Finally, the share of an object consumed by an agent is interpreted as the probability of

We borrow the notation of Kojima and Manea (2010) to formally define the PS rule. Given \( P \in \mathbb{D}^n \), for any \( a \in A' \subseteq A \), let \( N(a, A') \equiv \{ i \in I : a P_i b \text{ for all } b \in A' \setminus \{a\} \} \) be the set of agents whose favorite object in \( A' \) is \( a \).

**Definition 1** The Probabilistic Serial rule is a mapping \( PS : \mathbb{D}^n \rightarrow \mathcal{L} \), where given \( P \in \mathbb{D}^n \), the random assignment \( PS(P) \) is specified by the following iteration.

Initially, let \( t^0 \equiv 0, A^0 \equiv A, L^0_{ia} \equiv 0 \) for all \( i \in I \) and \( a \in A \).

For each \( v = 1, \ldots, \bar{v} \), let

\[
t^v \equiv \min_{a \in A^{v-1}} \max_{i \in I} \left\{ t \in [0, 1] : \sum_{i \in I} L_{ia}^{v-1} + |N(a, A^{v-1})| \cdot (t - t^{v-1}) \leq 1 \right\},
\]

\[
A^v \equiv A^{v-1} \setminus \left\{ a \in A^{v-1} : \sum_{i \in I} L_{ia}^{v-1} + |N(a, A^{v-1})| \cdot (t^v - t^{v-1}) = 1 \right\},
\]

\[
L_{ia}^v \equiv \begin{cases} 
L_{ia}^{v-1} + (t^v - t^{v-1}) & \text{if } i \in N(a, A^{v-1}), \\
L_{ia}^{v-1} & \text{otherwise.}
\end{cases}
\]

The final step \( \bar{v} \) is identified by \( A^{\bar{v}} = \emptyset \) and \( A^{\bar{v}-1} \neq \emptyset \). Let \( PS(P) \equiv \{ L_{ia}^v \}_{i \in I, a \in A} \).

For each period \( 1 \leq v \leq \bar{v} \), \( t^{v-1} \) denotes the beginning of this period, \( A^{v-1} \) denotes the set of available objects, and \( L^{v-1} \equiv \{ L_{ia}^{v-1} \}_{i \in I, a \in A} \) denotes the cumulative assignment. The end of the \( v \)-th period \( t^v \) is determined by the earliest time when some available object reaches its exhaustion, as defined by Equation (1). Then, we update the set of available objects \( A^v \) by excluding the exhausted objects from \( A^{v-1} \), as shown in Equation (2). Last, according to Equation (3), we update the cumulative assignment to \( L^v \equiv \{ L_{ia}^v \}_{i \in I, a \in A} \) by assigning to each agent \( t^v - t^{v-1} \) share of her favorite object in \( A^{v-1} \).

### 2.2 Weakly Connected Domains

We restrict attention to a large class of preference domains, weakly connected domains. Formally, two distinct preferences \( P_i, P_i' \) are called **neighbors**, denoted \( P_i \approx P_i' \), if whenever two objects are oppositely ranked across \( P_i \) and \( P_i' \), they are contiguously ranked at both \( P_i \) and \( P_i' \), i.e.,

\[
[a \ P_i b \text{ and } b \ P_i' a] \Rightarrow [a \ P_i! b \text{ and } b \ P_i'! a].
\]

Note that across two neighbored preferences, no object is involved in more than one preference reversal. Therefore, two neighbored preferences differ in locally switching several pair(s) of contiguously ranked objects, i.e., there exist \( \{(b_l, a_l) : l = 1, \ldots, t\} \), \( t \geq 1 \), and \( 1 \leq k_l < k_l + 1 < k_2 + 1 < \cdots < k_{t-1} + 1 < k_t < n \) such that (i) \( b_l = r_{k_l}(P_i) = r_{k_l+1}(P_i') \) and \( a_l = r_{k_l+1}(P_i) = r_{k_l}(P_i') \) for all \( l = 1, \ldots, t \), and (ii) \([x \ P_i y] \Leftrightarrow [x \ P_i' y]\) for all \((x, y) \not\in \{(b_l, a_l) : l = 1, \ldots, t\}\). In particular, if two neighbored preferences \( P_i \) and \( P_i' \) disagree on **exactly** one contiguously ranked pair of objects, they are called **adjacent**. We provide the following example to illustrate.
Example 1 Let preferences $P_1$, $P_2$ and $P_3$ be specified as below.

$$
\begin{align*}
P_1 : & \quad a \succ c \succ b \succ d \\
P_2 : & \quad a \succ b \succ c \succ d \\
P_3 : & \quad b \succ a \succ d \succ c
\end{align*}
$$

It is evident that $P_1$ and $P_2$ are adjacent, and $P_2$ and $P_3$ are neighbors, while $P_1$ and $P_3$ have no neighborhood relation. \hfill \square

A domain $\mathbb{D}$ is weakly connected if for all distinct $P_i, P'_i \in \mathbb{D}$, the difference of $P_i$ and $P'_i$ can be reconciled via local switchings along a sequence of neighbored preferences in the domain, i.e., there exists a path $\{P_i^k\}_{k=1}^{t} \subseteq \mathbb{D}$ connecting $P_i$ and $P'_i$ such that $P_i^1 = P_i$, $P_i^t = P'_i$, and $P_i^k \approx P_i^{k+1}$ for all $k = 1, \ldots, t-1$.

Remark 1 The class of weakly connected domains includes various instances that are widely studied in the ordinal mechanism design literature (including random assignment models) under both the one-dimensional and multidimensional settings: the universal domain (Gibbard, 1977; Abdulkadiroğlu and Sönmez, 1998), the single-peaked domain (Moulin, 1980), the single-dipped domain (Barberà et al., 2012), maximal single-crossing domains (Saporiti, 2009), the separable domain (Le Breton and Sen, 1999), the top-separable domain (Le Breton and Weymark, 1999) and the multidimensional single-peaked domain (Barberà et al., 1993). The notion of weak connectedness has also been extensively investigated in the literature of Condorcet domains (e.g., Monjardet, 2009; Puppe, 2018). Recently, some papers (e.g., Carroll, 2012; Sato, 2013; Cho, 2016a) study a proper subset of weakly connected domains, and show that, to ensure sd-SP, it suffices to guarantee that misreporting a preference adjacent to the sincere one is not profitable. \hfill \square

3 Main Results

It is well known that there exists no sd-SP, sd-Eff and ETE rule on the universal domain. We in this section investigate the preference restriction which restores the compatibility of these axioms. First, we use an example to intuitively show a preference restriction, and explain how it helps to ensure sd-SP of the PS rule. We then formally introduce our domain restriction, and prove that it is a necessary condition for the existence of an sd-SP, sd-Eff and ETE rule in the class of weakly connected domains. Last, we characterize the PS rule on our restricted domain via either sd-SP, sd-Eff and ETE, or sd-Eff and sd-EF.

Example 2 Let $A \equiv \{a, b, c, d\}$. Given preference profiles $P \equiv (P_1, P_2, P_3, P_4)$ and $P' \equiv (P_1, P_2, P_3, P'_4)$, we specify the PS assignments $PS(P)$ and $PS(P')$ below.

$$
\begin{align*}
P_1 : a & \succ c \succ b \succ d & PS(P) : & a & b & c & d \\
P_2 : a & \succ b \succ c \succ d & & 1 : 1/2 & 0 & 1/4 & 1/4 \\
P_3 : b & \succ a \succ c \succ d & & 2 : 1/2 & 0 & 1/4 & 1/4 \\
P_4 : b & \succ a \succ c \succ d & & 3 : 0 & 1/2 & 1/4 & 1/4 \\
P'_4 : a & \succ b \succ c \succ d & & 4 : 0 & 1/2 & 1/4 & 1/4
\end{align*}
$$

It reveals that the PS rule is not sd-SP: $PS_{4a}(P) + PS_{4b}(P) = \frac{1}{2} < \frac{5}{9} = PS_{4a}(P') + PS_{4b}(P')$ which says that by misreporting preference $P'_4$, agent 4 gets a higher probability of receiving an

\footnote{We put all formal definitions and the detailed verification in Appendix A.}
object strictly better than \( c \) in her true preference \( P_4 \). This manipulation occurs because the eating procedure of the PS rule is sensitive to deviations. In particular, agent 4’s misrepresentation (from \( P_4 \) to \( P'_4 \)) makes \( a \) reach its exhaustion earlier (from \( \frac{1}{2} \) to \( \frac{1}{5} \)). Notice that agent 1 prefers \( c \) to \( b \), while all others prefer both \( a \) and \( b \) to \( c \). Thus, from \( P \) to \( P' \), agent 1 starts to consume \( c \) earlier (from \( \frac{1}{2} \) to \( \frac{1}{5} \)), and hence consumes less \( a \) and \( b \) in total (from \( \frac{1}{2} \) to \( \frac{1}{5} \)). Consequently, agent 4 together with 2 and 3 consume more \( a \) and \( b \) in total (from \( \frac{1}{2} \) to \( \frac{5}{9} \)).

Next, we impose a tier-structure restriction on all agents’ preferences: Objects \( a \) and \( b \) always occupy the top two ranking positions. Thus, preference \( P_4 \) is no longer admissible. For instance, consider two other profiles \( \bar{P} \equiv (P_1, P_2, P_3, P_4) \) and \( \bar{P}' \equiv (P_1, P_2, P_3, P'_4) \), and the corresponding PS assignments below.

\[
\begin{array}{cccccc}
\bar{P}_1: & a \succ b \succ c \succ d & \quad & PS(\bar{P}): & a & b & c & d \\
\bar{P}_2: & a \succ b \succ c \succ d & \quad & 1: & 1/2 & 0 & 1/4 & 1/4 \\
\bar{P}_3: & b \succ a \succ c \succ d & \quad & 2: & 1/2 & 0 & 1/4 & 1/4 \\
\bar{P}_4: & b \succ a \succ c \succ d & \quad & 3: & 0 & 1/2 & 1/4 & 1/4 \\
\bar{P}_4': & a \succ b \succ c \succ d & \quad & 4: & 0 & 1/2 & 1/4 & 1/4 \\
\end{array}
\]

It turns out that agent 4’s misrepresentation is no longer profitable. Due to this particular tier structure, the combined probability of \( a \) and \( b \) assigned to agent 4 is fixed to \( \frac{1}{2} \) at both \( \bar{P}' \) and \( \bar{P} \). Consequently, the switch of \( a \) and \( b \) across \( P_4 \) and \( P'_4 \) makes agent 4 worse off as he consumes less of \( b \) at \( \bar{P}' \), i.e., \( PS_{4b}(\bar{P}') = \frac{1}{6} < \frac{1}{2} = PS_{4b}(\bar{P}) \). We therefore assert that the PS rule becomes \( sd-SP \).

Now, we formally introduce our preference restrictions. Let \( \mathcal{P} \equiv \{A_k\}_{k=1}^T \) denote a tier structure, i.e., (i) block \( A_k \subseteq A \) is nonempty for all \( k = 1, \ldots, T \), (ii) \( A_k \cap A_{k'} = \emptyset \) for all \( k \neq k' \), (iii) \( \bigcup_{k=1}^T A_k = A \), and (iv) \( \mathcal{P} \) is a linear order over all blocks. Next, we impose an additional restriction to define a restricted tier structure: Every block contains at most two objects, i.e., \( 1 \leq |A_k| \leq 2 \) for all \( k = 1, \ldots, T \). Then, we establish a (restricted) tier domain by requiring that the order of blocks in a tier structure be embedded in all preferences.

**Definition 2** A domain \( \mathcal{D} \) is a tier domain if there exists a tier structure \( \mathcal{P} \equiv \{A_k\}_{k=1}^T \) such that for all \( P_i \in \mathcal{D} \) and \( a, b \in A \), \([a \in A_k, b \in A_{k'} \text{ and } k < k'] \Rightarrow [a P_i b]\). Let \( \mathcal{D}(\mathcal{P}) \) denote the tier domain containing all admissible preferences. In particular, \( \mathcal{D} \subseteq \mathcal{D}(\mathcal{P}) \) is a restricted tier domain if \( \mathcal{P} \) is a restricted tier structure.

Note that all preferences of a restricted tier domain are pairwise neighbors. Hence, each restricted tier domain is weakly connected.

**Remark 2** In an auction model, Bikhchandani et al. (2006) study a class of domains with a particular tier structure, the order-based domains, where all (quasi linear) cardinal utilities induce an identical ordinal preference on objects at each payment level. More recently, domains with tier structures are also investigated in two-sided matchings (e.g., Akahoshi, 2014; Kandori et al., 2010), school choice (e.g., Kesten, 2010; Kesten and Kurino, 2013), and spectrum license auctions (Zhou and Serizawa, 2018).

\footnote{For notational convenience, we here simply write \( \mathcal{P} \) in an identity order.}
Remark 3 Notice that the restricted tier structure is a straightforward instance satisfying the value-restriction condition of Sen (1966), which is a sufficient condition for majority voting to be well-defined. A domain is value-restricted if for any three objects, one of them is never ranked either the best (among these three objects) in all preferences, or the medium in all preferences, or the worst in all preferences. □

Now, we present our first main result.

Theorem 1 If a weakly connected domain admits an sd-SP, sd-Eff and ETE rule, it is a restricted tier domain.

Proof: We prove Theorem 1 by two lemmas. Lemma 1 identifies two independent properties of an arbitrary domain (unnecessarily weakly connected), each of which implies the nonexistence of an sd-SP, sd-Eff and ETE rule. The first is called the local elevating property, while the second is called the double elevating property. Lemma 2 then pins down the restricted tier structure embedded in a weakly connected domain by cautiously avoiding the both properties.

We first introduce the local elevating property by a table of three preferences.

<table>
<thead>
<tr>
<th>Ranking:</th>
<th>$k$</th>
<th>$k + 1$</th>
<th>$k + 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_i$:</td>
<td></td>
<td>$a$</td>
<td>$b$</td>
</tr>
<tr>
<td>$B(P_i,a) = B(P_i,b)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$P_i$:</td>
<td></td>
<td>$a$</td>
<td>$c$</td>
</tr>
<tr>
<td>$B(P_i,a) = B(P_i,b)$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\hat{P}_i$:</td>
<td></td>
<td>$b$</td>
<td>$a$</td>
</tr>
<tr>
<td>$B(\hat{P}_i,a) = B(\hat{P}_i,b)$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1: The local elevating property

Observe first that, $\bar{P}_i$, $P_i$ and $\hat{P}_i$ share an identical set of top $(k - 1)$-ranked objects, and may differ in the rankings inside the identical set. Second, in all three preferences of Table 1, three objects $a$, $b$ and $c$ cluster in three consecutive ranking positions. Last, object $b$ takes three distinct positions while the relative ranking between $a$ and $c$ is fixed. From $\bar{P}_i$ to $P_i$, object $b$ is raised from the $(k + 2)$-th position to the $(k + 1)$-th position by locally overtaking $c$, while from $P_i$ to $\hat{P}_i$, $b$ is lifted one position further by locally overtaking $a$. Note that $P_1$, $P_2$ and $P_3$ of Example 2 satisfy the local elevating property. We formally introduce the definition of the local elevating property below.

Definition 3 Domain $\mathbb{D}$ satisfies the local elevating property if there exist $\bar{P}_i$, $P_i$, $\hat{P}_i \in \mathbb{D}$, $a$, $b$, $c \in A$ and $1 \leq k \leq n - 2$ satisfying the following two conditions:

1. $r_k(\bar{P}_i) = a$, $r_{k+1}(\bar{P}_i) = c$, $r_{k+2}(\bar{P}_i) = b$, $r_k(P_i) = a$, $r_{k+1}(P_i) = b$, $r_{k+2}(P_i) = c$, $r_k(\hat{P}_i) = b$, $r_{k+1}(\hat{P}_i) = a$, $r_{k+2}(\hat{P}_i) = c$, and

2. $B(\bar{P}_i,a) = B(P_i,a) = B(\hat{P}_i,b)$.

The double elevating property differs from the local elevating property as we introduce an additional object $d$ which is consecutively ranked below $c$ in $P_i$, and ranks above $c$ in $\hat{P}_i$. Consequently, besides the same local elevating process of object $b$ in Table 1, one would observe an additional elevating process in the opposite direction: Object $c$ overtakes $d$ from the $(k + 3)$-th position at $\hat{P}_i$ to the $(k + 2)$-th position at $P_i$, and continues to overtake $b$ from $P_i$ to $\hat{P}_i$ (see Table 2 below).
\( \begin{array}{c|c|c|c|c} k & k+1 & k+2 & k+3 \\ \hline \end{array} \)

\( \hat{P}_i: \quad \cdots \cdots \cdots \quad \succ a \succ c \succ b \succ \cdots \cdots \)
\( B(\hat{P}_i,a) = B(\hat{P}_i,a) \)

\( P_i: \quad \cdots \cdots \cdots \quad \succ a \succ b \succ c \succ d \succ \cdots \)
\( B(P_i,a) = B(P_i,b) \)

\( \hat{P}_i: \quad \cdots \cdots \cdots \quad \succ b \succ a \succ d \succ c \succ \cdots \)

Table 2: The double elevating property

**Definition 4** Domain \( \mathbb{D} \) satisfies the **double elevating property** if there exist \( \hat{P}_i, P_i, \hat{P}_i \in \mathbb{D}, a, b, c, d \in A, \) and \( 1 \leq k \leq n - 3 \) satisfying the following two conditions:

1. \( r_k(\hat{P}_i) = a, r_{k+1}(\hat{P}_i) = c, r_{k+2}(\hat{P}_i) = b, r_{k+3}(\hat{P}_i) = d, \)
2. \( B(\hat{P}_i,a) = B(P_i,a) = B(\hat{P}_i,b). \)

**Lemma 1** A domain satisfying the local elevating property or the double elevating property admits no \( sd-SP, sd-Eff \) and \( ETE \) rule.

The proof of Lemma 1 is put in Appendix B. We provide here an outline of the proof strategy. Let domain \( \mathbb{D} \) satisfy the local elevating property.\(^6\) Suppose that there exists an \( sd-SP, sd-Eff \) and \( ETE \) rule \( \varphi : \mathbb{D}^n \to \mathcal{L}. \) We illustrate below how a contradiction is identified in the case where \( n \) is even.\(^7\) We first construct the following two preference profiles, which involve only the preferences in Table 1.

- \( P^{3,2} \equiv (\hat{P}_1, \ldots, \hat{P}_{n-1}, P_{\frac{n}{2}}, P_{\frac{n}{2}+1}, \ldots, P_{n-1}, \hat{P}_n): \) Agents 1, \ldots, \( \frac{n}{2} \) report preference \( \hat{P}_i, \)
  agents \( \frac{n}{2} + 1, \ldots, n - 1 \) report \( P_i, \) while agent \( n \) reports \( \hat{P}_i. \)
- \( P^{4,2} \equiv (\hat{P}_1, \ldots, \hat{P}_{n-1}, \hat{P}_{\frac{n}{2}}, P_{\frac{n}{2}+1}, \ldots, P_{n-1}, \hat{P}_n): \) Agents 1, \ldots, \( \frac{n}{2} \) report preference \( \hat{P}_i, \)
  agents \( \frac{n}{2} + 1, \ldots, n - 1 \) report \( P_i, \) while agent \( n \) reports \( \hat{P}_i. \)

Note that \( P^{3,2} \) and \( P^{4,2} \) differ exactly in agent \( \frac{n}{2} \)'s preferences, i.e., \( P^{3,2}_{\frac{n}{2}} = P_i \) and \( P^{4,2}_{\frac{n}{2}} = \hat{P}_i. \) Then, \( sd-SP \) requires \( \varphi_{\frac{n}{2}}(P^{3,2}) + \varphi_{\frac{n}{2}}(P^{3,2}) = \varphi_{\frac{n}{2}}(P^{4,2}) + \varphi_{\frac{n}{2}}(P^{4,2}). \) We will induce a contradiction where this equality does not hold. In order to do so, we investigate two sequences of preference profiles.

The first sequence starts from \( (P_1, P_2, \ldots, P_{n-1}, \hat{P}_n), \) and gradually changes to \( P^{3,2} \) by switching, one by one, the preferences of agents \( 1, 2, \ldots, \frac{n}{2} - 1 \) from \( P_i \) to \( \hat{P}_i. \) For each profile of this sequence, we characterize the probabilities of \( a \) and \( b. \) Eventually, we determine \( \varphi_{\frac{n}{2}}(P^{3,2}_{\frac{n}{2}}) + \varphi_{\frac{n}{2}}(P^{3,2}_{\frac{n}{2}}). \) The second sequence goes from \( (\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_{n-1}, P_n) \) to \( P^{4,2}_{\frac{n}{2}}, \) and changes, one by one, agents \( n - 1, n - 2, \ldots, \frac{n}{2} + 1 \)’s preferences from \( \hat{P}_i \) to \( P_i. \) For each preference profile in this sequence, we also characterize the probabilities of \( a \) and \( b. \) Eventually, we determine \( \varphi_{\frac{n}{2}}(P^{4,2}_{\frac{n}{2}}) + \varphi_{\frac{n}{2}}(P^{4,2}_{\frac{n}{2}}), \) and verify that it is different from \( \varphi_{\frac{n}{2}}(P^{3,2}_{\frac{n}{2}}) + \varphi_{\frac{n}{2}}(P^{3,2}_{\frac{n}{2}}). \)

\(^6\)When the domain satisfies instead the double elevating property, the same proof strategy applies.
\(^7\)For the case of an odd number of agents, a similar contradiction can be identified with some extra effort.
Now, according to Lemma 1, the hypothesis of Theorem 1 implies that the weakly connected domain in question must violate both the local elevating and double elevating properties. The next lemma utilizes the negation of both properties to elicit the embedded restricted tier structure.

**Lemma 2** A weakly connected domain avoiding both the local elevating and double elevating properties is a restricted tier domain.

The proof of Lemma 2 is put in Appendix C. We provide here a proof outline. Let \( D \) be a weakly connected domain, and violate both the local elevating and double elevating properties. We first show that among any three distinct preferences of \( D \), if two pairs of them are neighbors, then all three are pairwise neighbors. Consequently, by weak connectedness, since every pair of distinct preferences is connected via a path, they must be neighbors. Therefore, all preferences of \( D \) are pairwise neighbors. This implies that \( D \) must be a restricted tier domain.

In conclusion, combining Lemmas 1 and 2, we complete the proof of Theorem 1.

**Remark 4** Theorem 1 still holds when \( |A| \neq |I| \). When \( |A| > |I| \geq 3 \), Lemma 1 holds by arbitrarily choosing \( |A| - |I| \) objects as the commonly least preferred objects. When \( |I| > |A| \geq 4 \), Lemma 1 still holds by introducing \( |I| - |A| \) new objects as the commonly least preferred objects. In addition, Lemma 2 holds no matter whether \( |A| = |I| \) or not since it pins down the restricted tier structure using only weak connectedness and the negation of both the local elevating and double elevating properties.

**Remark 5** Our Lemma 1 generalizes the recent impossibility theorem of Chang and Chun (2017) which says that there is no sd-SP, sd-Eff and ETE rule on a domain including three particular preferences such that one object takes the bottom three ranking positions respectively while all the other objects are identically ranked. Therefore, their preference condition is in fact a special case of our local elevating property. The proof strategy of Chang and Chun (2017) is applicable for showing the impossibility under the local elevating property, but becomes invalid under the double elevating property. More importantly, our more stylized pattern of the local elevating and double elevating properties allows us to pin down the restricted tier structure.

**Remark 6** If we strengthen the fairness axiom from ETE to sd-EF, the proof of Lemma 1 can be significantly simplified (see Appendix D).

The next result characterizes the PS rule as the unique desirable rule on a restricted tier domain.

**Theorem 2** Let \( D \) be a restricted tier domain. Fix a rule \( \varphi : D^n \to L \). The following three statements are equivalent:

(i) \( \varphi \) is sd-SP, sd-Eff and ETE.

(ii) \( \varphi \) is sd-Eff and sd-EF.

(iii) \( \varphi \) is the PS rule.

**Proof:** Let \( \mathcal{P} \equiv (A_k)_{k=1}^T \) be a restricted tier structure, and \( D \subseteq D(\mathcal{P}) \). We prove the theorem by two steps. The first proves the equivalence between (i) and (iii), and the second shows the equivalence between (ii) and (iii). Before these two steps, we present the following fact, which observes that, due to the restricted tier structure embedded in domain \( D \), the definition of the PS rule (Definition 1) is significantly simplified.
Fact 1 Given $P \in \mathbb{D}^n$, an assignment $L = PS(P)$ if and only if the following two conditions hold for every block $A_k, k = 1, \ldots, T$:

1. If $A_k \equiv \{a\}$, we have $L_{ia} = \frac{1}{n}$ for all $i \in I$.

2. If $A_k \equiv \{a, b\}$, let $I_k \equiv \{i \in I : a \ P_i \ b\}$, and we have $L$ over $A_k$ specified below:

$$
\begin{bmatrix}
|I_k| \geq \frac{n}{2} \\
|I_k| \leq \frac{n}{2}
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
i \in I_k : & a & b \\
j \notin I_k : & 0 & 2 - \frac{1}{|I_k|}
\end{bmatrix},
$$

and

$$
\begin{bmatrix}
i \in I_k : & \frac{a}{2} & \frac{b}{2} \\
j \notin I_k : & 0 & 0
\end{bmatrix}.
$$

Condition 1 says that if a block contains exactly one object, all agents equally share it. Condition 2 specifies the assignment of two objects in the same block, say $A_k = \{a, b\}$. The set $I_k \equiv \{i \in I : a \ P_i \ b\}$ is either the majority group (if $|I_k| \geq \frac{n}{2}$), or the minority group (if $|I_k| \leq \frac{n}{2}$). If $I_k$ is the majority group, each agent of them consumes $\frac{1}{|I_k|}$ share of $a$ and $\frac{2}{n} - \frac{1}{|I_k|}$ of $b$, while each agent of $I \backslash I_k$ (provided $I \backslash I_k \neq \emptyset$) only consumes $\frac{2}{n}$ of $b$. If $I_k$ is the minority group, the symmetric case applies.

To see that the random assignment specified above is exactly the PS assignment of Definition 1, recall the eating procedure applied on a profile $P$ of a restricted tier domain. If the top ranked block contains exactly one object, every agent will consume $\frac{1}{n}$ share of it. Otherwise, let the top ranked block $A_1 \equiv \{a, b\}$. Then, all agents of $I_1 \equiv \{i \in I : a \ P_i \ b\}$ will consume $a$, and others will consume $b$ until one object of $A_1$ reaches its exhaustion, or both reach the exhaustion simultaneously. Clearly, which object will be exhausted earlier depends on the relative size of $I_1$ and $I \backslash I_1$. Specifically, according to Definition 1, we have the following observations:

- If $|I_1| = \frac{n}{2}$, then $t^1 = \frac{2}{n}$, $A^1 = A \backslash \{a, b\}$ and $L^1 = \begin{bmatrix} a & b & A \backslash \{a, b\} \\ i \in I_1 : & \frac{2}{n} & 0 & 0 \\ j \notin I_1 : & 0 & \frac{2}{n} & 0 \end{bmatrix}$.

- If $|I_1| > \frac{n}{2}$, then $t^1 = \frac{1}{|I_1|}$, $A^1 = A \backslash \{a\}$ and $L^1 = \begin{bmatrix} a & b & A \backslash \{a, b\} \\ i \in I_1 : & \frac{1}{|I_1|} & 0 & 0 \\ j \notin I_1 : & 0 & \frac{1}{|I_1|} & 0 \end{bmatrix}$.

- If $|I_1| < \frac{n}{2}$, then $t^1 = \frac{1}{n-|I_1|}$, $A^1 = A \backslash \{b\}$ and $L^1 = \begin{bmatrix} a & b & A \backslash \{a, b\} \\ i \in I_1 : & \frac{1}{n-|I_1|} & 0 & 0 \\ j \notin I_1 : & 0 & \frac{1}{n-|I_1|} & 0 \end{bmatrix}$.

In words, if $t^1 = \frac{2}{n}$, then the whole block $A_1$ is exhausted at time $t^1$. If $t^1 \neq \frac{2}{n}$, all agents will consume the remaining object in $A_1$ after time $t^1$ until the exhaustion of the whole block $A_1$. Then, according to Definition 1, we have $t^2 = \frac{2}{n}$ and $A^2 = A \backslash \{a, b\}$, and update the assignment to $L^2$ below:

$$
\begin{bmatrix}
|I_1| > \frac{n}{2} \\
|I_1| < \frac{n}{2}
\end{bmatrix}
\Rightarrow
L^2 = \begin{bmatrix} a & b & A \backslash \{a, b\} \\ i \in I_1 : & \frac{1}{|I_1|} & \frac{2}{n} - \frac{1}{|I_1|} & 0 \\ j \notin I_1 : & 0 & \frac{2}{n} & 0 \end{bmatrix},
$$

and

$$
\begin{bmatrix}
|I_1| \leq \frac{n}{2} \\
|I_1| > \frac{n}{2}
\end{bmatrix}
\Rightarrow
L^2 = \begin{bmatrix} a & b & A \backslash \{a, b\} \\ i \in I_1 : & \frac{2}{n} & 0 & 0 \\ j \notin I_1 : & \frac{2}{n} - \frac{1}{n-|I_1|} & \frac{1}{n-|I_1|} & 0 \end{bmatrix}.
$$
Notice that every agent consumes $\frac{2}{n}$ of $a$ and $b$ in combination. By examining the eating procedure on all blocks consecutively, we eventually obtain the PS assignment as specified in Fact 1. Thus, in the following two steps, we refer to Fact 1 as the definition of the PS rule.

**Step 1:** (i) $\iff$ (iii)

As shown in Example 2, it is easy to verify that the PS rule is $sd$-$SP$ on domain $\mathbb{D}$. We next show (i) $\implies$ (iii). We fix an arbitrary profile $P \equiv (P_1, P_2, \ldots, P_n) \in \mathbb{D}^n$, and show that $\varphi(P)$ is exactly the one specified in Fact 1. The proof consists of three claims below.

**Claim 1** $\varphi_{iA_k}(P) = \frac{|A_k|}{n}$ for all $i \in I$ and $k = 1, \ldots, T$.

Fix arbitrary $\bar{P}_1 = \bar{P}_2 = \cdots = \bar{P}_n$. Let $P^0 = (\bar{P}_1, \bar{P}_2, \ldots, \bar{P}_n)$. We consider the following $n$ groups of preference profiles:

- **Group 1:** $P^{i} \equiv (P_i, \bar{P}_{-i})$ for each $i \in I$,
- $\vdots$
- **Group $l$:** $P^l \equiv (P_I, \bar{P}_{-I})$ for each $I \subseteq I$ with $|I| = l$,
- $\vdots$
- **Group $n$:** $P^n \equiv (P_1, P_2, \ldots, P_n)$.

Note that for each $1 \leq l \leq n$, group $l$ contains $\frac{n!}{l!(n-l)!}$ preference profiles, and $P^l = P$. We show that for each group $1 \leq l \leq n$ and each profile $P^l$ of group $l$, $\varphi_{iA_k}(P^l) = \frac{|A_k|}{n}$ for all $i \in I$ and $k = 1, \ldots, T$.

It is evident that $ETE$ and feasibility imply $\varphi_{iA_k}(P^0) = \frac{|A_k|}{n}$ for all $i \in I$ and $k = 1, \ldots, T$. We next provide an induction hypothesis: Given $0 < l \leq n$, for every preference profile $P^l$ of group $l-1$, we have $\varphi_{iA_k}(P^l) = \frac{|A_k|}{n}$ for all $i \in I$ and $k = 1, \ldots, T$. Given an arbitrary $\bar{I} \subseteq I$, let $|\bar{I}| = l$, and we show $\varphi_{iA_k}(P^l) = \frac{|A_k|}{n}$ for all $i \in I$ and $k = 1, \ldots, T$.

Note that $\bar{I} \neq \emptyset$. For notational convenience, we assume w.l.o.g. that $\bar{I} = \{1, \ldots, l\}$. Given an arbitrary $i \in \bar{I}$, let $\bar{I} = \bar{I}\backslash\{i\}$. From $P^l \equiv (P_1, \ldots, P_{i-1}, P_i, P_{i+1}, \ldots, P_{l}, \bar{P}_{-j})$ to $P^l \equiv (P_1, \ldots, P_{i-1}, \bar{P}_i, P_{i+1}, \ldots, P_{l}, \bar{P}_{-j})$, agent $i$ unilaterally deviates from $P_i$ to $\bar{P}_i$. Since $P_i$ and $\bar{P}_i$ share the same ranking over all blocks $A_1, \ldots, A_T$, $sd$-$SP$ and the induction hypothesis imply $\varphi_{iA_k}(P^l) = \varphi_{iA_k}(P^l) = \frac{|A_k|}{n}$ for all $k = 1, \ldots, T$. Therefore, we know $\varphi_{iA_k}(P^l) = \frac{|A_k|}{n}$ for all $i \in \bar{I}$ and $k = 1, \ldots, T$. If $l = n$ (equivalently, $\bar{I} = I$), we have completed the verification of the induction hypothesis. Otherwise, since all agents of $I\backslash\bar{I}$ have the same preference, $ETE$ and feasibility imply $\varphi_{jA_k}(P^l) = \frac{|A_k| - \frac{|A_k|}{n}}{n - \frac{|A_k|}{n}} = \frac{|A_k|}{n}$ for all $j \in I\backslash\bar{I}$ and $k = 1, \ldots, T$. Therefore, $\varphi_{iA_k}(P^l) = \frac{|A_k|}{n}$ for all $i \in I$ and $k = 1, \ldots, T$. This completes the verification of the induction hypothesis, and hence proves the claim.

Thus, for each $1 \leq k \leq T$, if $A_k$ is a singleton set, say $A_k \equiv \{a\}$, we have $\varphi_{ia}(P) = \frac{1}{n}$ for all $i \in I$. Therefore, $\varphi(P)$ meets the first condition of Fact 1. Next, fix an arbitrary block $A_k \equiv \{a, b\}$. Let $I_k \equiv \{i \in I : a P_i b\}$ and $l \equiv |I_k|$. Assume w.l.o.g. that $l \geq \frac{n}{2}$. The verification related to $l \leq \frac{n}{2}$ is symmetric, and we hence omit it.

**Claim 2** $\varphi_{ja}(P) = 0$ and $\varphi_{jb}(P) = \frac{2}{n}$ for all $j \in I \backslash I_k$. 

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If \( l = n \), then \( I \setminus I_k = \emptyset \), and the claim holds vacuously. Next, assume \( l < n \). Thus, \( I \setminus I_k \neq \emptyset \). Suppose that there exists \( j^* \in I \setminus I_k \) such that \( \varphi_{j^* a}(P) > 0 \). Since each agent of \( I_k \) prefers \( a \) to \( b \), sd-Eff implies \( \varphi_{ib}(P) = 0 \) for all \( i \in I_k \). Then, feasibility implies \( \sum_{j \in I \setminus I_k} \varphi_{jb}(P) = 1 \). Thus, there are two cases to consider: (i) There exists \( j \in I \setminus I_k \) such that \( \varphi_{jb}(P) > \frac{1}{m-l} \), and (ii) \( \varphi_{jb}(P) = \frac{1}{m-l} \) for all \( j \in I \setminus I_k \). For case (i), \( \varphi_{jA_k}(P) = \varphi_{ja}(P) + \varphi_{jb}(P) > \frac{1}{m-l} \geq \frac{2}{n} \) which contradicts Claim 1. For case (ii), \( \varphi_{jA_k}(P) \equiv \varphi_{ja}(P) + \varphi_{jb}(P) > \frac{1}{m-l} \geq \frac{2}{n} \) which also contradicts Claim 1. Therefore, \( \varphi_{ja}(P) = 0 \) for all \( j \in I \setminus I_k \). Then, by Claim 1, we have \( \varphi_{jb}(P) = \frac{2}{n} \) for all \( j \in I \setminus I_k \). This completes the verification of the claim.

**Claim 3** \( \varphi_{ia}(P) = \frac{1}{t} \) and \( \varphi_{ib}(P) = \frac{2}{n} - \frac{1}{t} \) for all \( i \in I_k \).

Fix an arbitrary preference \( \tilde{P}_t \in \mathbb{D} \) with \( a \tilde{P}_t b \). We first construct preference profile \( P^0 = (\tilde{P}_k, P_{-I_k}) \) where every agent of \( I_k \) reports preference \( \tilde{P}_t \), and every agent \( j \in I \setminus I_k \) reports preference \( P_j \) in profile \( P \). To prove the claim, we consider the following \( l \) group of preference profiles:

- **Group 1:** \( P^{(i)} \equiv (P_i, \tilde{P}_{I_k \setminus \{i\}}, P_{-I_k}) \) for each \( i \in I_k \)
- **Group 1:** \( P^{(l)} \equiv (P_l, \tilde{P}_{I_k \setminus \{l\}}, P_{-I_k}) \) for each \( \hat{l} \subseteq I_k \) with \( |\hat{l}| = m \)
- **Group l:** \( P^{I_k} \equiv (P_k, P_{-I_k}) \)

Note that for each \( 1 \leq m \leq l \), group \( m \) contains \( \frac{l!}{m!(l-m)!} \) preference profiles, and \( P^{I_k} \equiv P \). We show that for each group \( 1 \leq m \leq l \) and each profile \( P^{I} \) of group \( m \), \( \varphi_{ia}(P^{I}) = \frac{1}{t} \) and \( \varphi_{ib}(P^{I}) = \frac{2}{n} - \frac{1}{t} \) for all \( i \in I_k \).

First, similar to Claim 2, we have \( \varphi_{ja}(P^0) = 0 \) and \( \varphi_{jb}(P^0) = \frac{2}{n} \) for all \( j \in I \setminus I_k \). Then, ETE and feasibility imply \( \varphi_{ia}(P^{0}) = \frac{1}{t} \) and \( \varphi_{ib}(P^{0}) = \frac{1}{t} \) for all \( i \in I_k \). Next, we adopt an induction hypothesis: Given \( 0 < m \leq l \), for every preference profile \( P^{I} \) of group \( m-1 \), we have \( \varphi_{ia}(P^{I}) = \frac{1}{t} \) and \( \varphi_{ib}(P^{I}) = \frac{2}{n} - \frac{1}{t} \) for all \( i \in I_k \). Given an arbitrary \( \hat{l} \subseteq I_k \), let \( |\hat{l}| = m \), and we show \( \varphi_{ia}(P^{I}) = \frac{1}{t} \) and \( \varphi_{ib}(P^{I}) = \frac{2}{n} - \frac{1}{t} \) for all \( i \in I_k \).

Note that \( \hat{l} \neq \emptyset \). For notational convenience, we assume w.l.o.g. that \( \hat{l} = \{1, \ldots, m\} \). Given an arbitrary \( i \in \hat{l} \), let \( \hat{I} = \hat{l} \setminus \{i\} \). From \( P^{I} \equiv (P_1, \ldots, P_{l-1}, P_1, P_{l+1}, \ldots, P_m, \tilde{P}_{I_k \setminus \{i\}}, P_{-I_k}) \) to \( P^{I} \equiv (P_1, \ldots, P_{l-1}, \tilde{P}_i, P_{l+1}, \ldots, P_m, \tilde{P}_{I_k \setminus \{i\}}, P_{-I_k}) \), agent \( i \) unilaterally deviates from \( P_i \) to \( \tilde{P}_i \). Since \( P_i \) and \( \tilde{P}_i \) share the same ranking over all blocks \( A_1, \ldots, A_T \), and both rank \( a \) over \( b \), sd-SP and the induction hypothesis imply \( \varphi_{ia}(P^{I}) = \varphi_{ia}(P^{I}) = \frac{1}{t} \) and \( \varphi_{ib}(P^{I}) = \varphi_{ib}(P^{I}) = \frac{2}{n} - \frac{1}{t} \). Therefore, we know \( \varphi_{ia}(P^{I}) = \frac{1}{t} \) and \( \varphi_{ib}(P^{I}) = \frac{2}{n} - \frac{1}{t} \) for all \( i \in \hat{l} \). If \( m = l \) (equivalently, \( \hat{l} = I_k \)), we have completed the verification of the induction hypothesis. Otherwise, we consider agents of \( I_k \setminus \hat{l} \) at profile \( P^{I} \). Similar to Claim 2, we know \( \varphi_{ja}(P^{I}) = 0 \) and \( \varphi_{jb}(P^{I}) = \frac{2}{n} \) for all \( j \in I \setminus I_k \). Then, ETE and feasibility imply \( \varphi_{ia}(P^{I}) = \frac{1-m}{l-m} \equiv \frac{1}{t} \) and \( \varphi_{ib}(P^{I}) = \frac{1-m}{l-m} \equiv \frac{2}{n} - \frac{1}{t} \) for all \( j \in I_k \). Therefore, \( \varphi_{ia}(P^{I}) = \frac{1}{t} \) and \( \varphi_{ib}(P^{I}) = \frac{2}{n} - \frac{1}{t} \) for all \( i \in I_k \). This completes the verification of the induction hypothesis, and hence proves the claim.

Thus, by Claims 2 and 3, \( \varphi(P) \) satisfies the second condition of Fact 1, as required. This completes the verification of Step 1.

**Step 2:** (ii) \( \Leftrightarrow \) (iii)
By Bogomolnaia and Moulin (2001), we know that the PS assignment at any preference profile is $sd$-$Eff$ and $sd$-$EF$. We show (ii) ⇒ (iii).\(^8\)

Fix an arbitrary profile $P \in \mathbb{D}^n$. First, since all preferences of $P$ share the same rankings over all blocks $A_1, \ldots, A_T$, $sd$-$EF$ and feasibility imply $\varphi_{iA_k}(P) = \frac{|A_k|}{n}$ for all $i \in I$ and $k = 1, \ldots, T$. Thus, for each $1 \leq k \leq T$, if $A_k$ is a singleton set, say $A_k \equiv \{a\}$, we have $\varphi_{ia}(P) = \frac{1}{n}$ for all $i \in I$. Therefore, $\varphi(P)$ meets condition 1 of Fact 1.

Second, fix an arbitrary block $A_k \equiv \{a, b\}$. Given $I_k \equiv \{i \in I : a \not\prec b\}$, we assume w.l.o.g. that $|I_k| = l \geq \frac{n}{2}$. The verification related to $|I_k| = l \leq \frac{n}{2}$ is symmetric, and we hence omit it. If $l = n$, then all agents prefer $a$ to $b$, and $sd$-$EF$ and feasibility imply $\varphi_{ia}(P) = \frac{1}{n} \equiv \frac{1}{l}$ and $\varphi_{ib}(P) = \frac{1}{n} \equiv \frac{2}{n} - \frac{1}{l}$ for all $i \in I$, which meet condition 2 of Fact 1. Furthermore, assume $l < n$. We assert $\varphi_{ja}(P) = 0$ for all $j \in I \setminus I_k$. Suppose not, i.e., there exists $j^* \in I \setminus I_k$ such that $\varphi_{j^*a}(P) > 0$. Since all agents of $I_k$ prefer $a$ to $b$, $sd$-$Eff$ implies $\varphi_{ib}(P) = 0$ for all $i \in I_k$. Then, $sd$-$EF$ and feasibility imply $\varphi_{jb}(P) = \frac{1}{N-I_k}$. Consequently, $\varphi_{j^*a}(P) \equiv \varphi_{j^*a}(P) + \varphi_{jb}(P) > \frac{1}{N-I_k} \geq \frac{2}{n}$. Contradiction! Therefore, $\varphi_{ja}(P) = 0$ for all $j \in I \setminus I_k$. Then, $sd$-$EF$ and feasibility imply $\varphi_{ia}(P) = \frac{1}{l}$ for all $i \in I_k$. Last, since $\varphi_{ia}(P) + \varphi_{ib}(P) = \frac{2}{n}$ for all $i \in I$, we have $\varphi_{ib}(P) = \frac{2}{n} - \frac{1}{l}$ for all $i \in I_k$, and $\varphi_{jb}(P) = \frac{2}{n}$ for all $j \in I \setminus I_k$. This proves the second condition of Fact 1 on $\varphi(P)$, and hence completes the verification of Step 2. ■

Remark 7 Theorem 2 still holds when $|I| \neq |A|$ and agents have outside options.\(^9\) In particular, a preference is defined as a linear order on $A \cup \{\emptyset\}$ where $\emptyset$ denotes an outside option, and we extend the definition of restricted tier domains in the following sense: (i) An admissible preference treats a certain number of top blocks as acceptable, i.e., better than the outside option, and (ii) an admissible preference is only required to respect a restricted tier structure over its acceptable blocks. This extended notion of restricted tier domains strictly nests the preference domain studied by Bogomolnaia and Moulin (2002), and therefore the extension of Theorem 2 implies their characterization results. \(\square\)

Remark 8 We notice that the random serial dictatorship rule remains $sd$-inefficient on a restricted tier domain. For instance, at the profile $\bar{P}$ of Example 2, the PS assignment Pareto dominates the random serial dictatorship assignment.

\[
\begin{array}{ccccccc}
\text{PS}(\bar{P}) : & a & b & c & d \\
1 : & 1/2 & 0 & 1/4 & 1/4 \\
2 : & 1/2 & 0 & 1/4 & 1/4 \\
3 : & 0 & 1/2 & 1/4 & 1/4 \\
4 : & 0 & 1/2 & 1/4 & 1/4 \\
\end{array}
\]

\[
\begin{array}{ccccccc}
\text{RSD}(\bar{P}) : & a & b & c & d \\
1 : & 5/12 & 1/12 & 1/4 & 1/4 \\
2 : & 5/12 & 1/12 & 1/4 & 1/4 \\
3 : & 1/12 & 5/12 & 1/4 & 1/4 \\
4 : & 1/12 & 5/12 & 1/4 & 1/4 \\
\end{array}
\]

\(\square\)

4 An Extension: Beyond Weak Connectedness

In this section, we investigate domains which are not weakly connected, and admit an $sd$-$SP$, $sd$-$Eff$ and $ETE$ rule. As mentioned in the introduction, Theorem 1 implies that any domain

\(\text{For simplicity, we provide a direct proof. Alternatively, due to the restricted tier structure, we observe that each preference profile satisfies the Type 1 and 3 decomposition conditions of Cho (2016b). Then, Theorem 3 of Cho (2016b) implies that the PS assignment at each preference profile is the unique one satisfying } sd$-$Eff$ $\text{and } sd$-$EF$. \(\text{See the details in the working paper version which is available at } http://ink.library.smu.edu.sg/soe_ research/1860/.)\)
(not necessarily weakly connected) admitting an sd-SP, sd-Eff and ETE rule must be a union of multiple restricted tier domains. Moreover, Lemma 1 implies that both the local elevating and double elevating properties must be violated. We introduce the notion of dichotomous refinement which systematically excludes preferences in order to reduce the instances of the local elevating property. We then provide an algorithm which repeatedly applies dichotomous refinements, and eventually generates a union of multiple restricted tier domains which completely avoids both the local elevating and double elevating properties (see Lemma 3). More importantly, we show that every domain generated by the algorithm is equivalent to a sequentially dichotomous domain of Liu (2018), and therefore restores sd-SP on the PS rule (see Proposition 1).

Given a tier structure $P = (A_1, \ldots, A_{t-1}, A_t, A_{t+1}, \ldots, A_T)$, we say that two tier structures $\overline{P}$ and $\overline{P}$ are the dichotomous refinements of $P$ if exactly one block $A_i$ breaks into two nonempty subsets $A_i^1$ and $A_i^2$, i.e., $A_i^1 \cap A_i^2 = \emptyset$ and $A_i^1 \cup A_i^2 = A_i$, such that

$$\overline{P} = (A_1, \ldots, A_{t-1}, A_i^1, A_i^2, A_{t+1}, \ldots, A_T) \quad \text{and} \quad \overline{P} = (A_1, \ldots, A_{t-1}, A_i^2, A_i^1, A_{t+1}, \ldots, A_T).$$

In particular, when $|A_t| \geq 3$, via dichotomous refinements, some preferences in the original tier domain $D(P)$ are excluded from the refined tier domains $D(\overline{P})$ and $D(\overline{P})$, which turn out to reduce instances of the local elevating property. We present in the following example to illustrate. One can easily see that the dichotomous refinements also reduce instances of the double elevating property when $|A_t| \geq 4$.

**Example 3** Let $A \equiv \{a, b, c, d\}$ and $P \equiv (\{a\}, \{b, c, d\})$ be a tier structure. We break the block $\{b, c, d\}$ into $\{b, c\}$ and $\{d\}$, and induce the dichotomous refinements $\overline{P} = (\{a\}, \{b, c\}, \{d\})$ and $\overline{P} = (\{a\}, \{d\}, \{b, c\})$. When $D(P)$ shrinks to $D(\overline{P}) \cup D(\overline{P})$, the instance of the local elevating property below which appeared in $D(P)$ is eliminated because $\overline{P}$ is excluded.

\[
\begin{align*}
\hat{P}_2: & \quad a \succ c \succ d \succ b \\
\hat{P}_2: & \quad a \succ c \succ b \succ d \\
\hat{P}_2: & \quad a \succ b \succ c \succ d \\
\end{align*}
\]

Now, we present the algorithm to repeatedly exclude preferences via a sequence of dichotomous refinements, and eventually generate a union of restricted tier domains. We first introduce the notation used in the algorithm. Given a tier structure $P = (A_1, \ldots, A_T)$, let $A = \{A_1, \ldots, A_T\}$ denote the partition of $A$ which collects all blocks in $P$.

**Algorithm:** Initially, set $\Omega_0 \equiv (A)$, $A_0 \equiv \{A\}$ and $D(\Omega_0) \equiv P$.

**Step 1.** Fixing an arbitrary nonempty proper subset $\tilde{A} \subset A$, let

- $\Omega_1 \equiv \{ (A, A \setminus \tilde{A}), (A \setminus \tilde{A}, A) \}$,
- $D(\Omega_1) \equiv \bigcup_{P \in \Omega_1} D(P)$ denote the union of corresponding tier domains, and
- $A_1 \equiv \{ \tilde{A}, A \setminus \tilde{A} \}$ be the corresponding partition of $A$.

If there exists a block of $A_1$ containing more than 2 objects, proceed to the next step. Otherwise, terminate the algorithm.

---

$^{10}$Formally, a partition of $A$ is a collection of mutually exclusive and exhaustive nonempty subsets of $A$. Blocks are not ordered in a partition, but linearly ordered in a tier structure.
Step $k > 1$. Let $A_t \in A_{k-1}$ be an arbitrary block such that $|A_t| > 2$. Break $A_t$ into two nonempty subsets $A^1_t$ and $A^2_t$. Then, let

- $\Omega_k \equiv \bigcup_{\mathcal{P} \in \Omega_{k-1}} \{ \overline{\mathcal{P}}, \overline{\mathcal{P}} : \overline{\mathcal{P}}$ and $\mathcal{P}$ are the corresponding dichotomous refinements of $\mathcal{P} \}$,
- $\mathbb{D}(\Omega_k) \equiv \bigcup_{\mathcal{P} \in \Omega_k} \mathbb{D}(\mathcal{P})$ denote the union of corresponding tier domains, and
- $A_k \equiv \{A^1_t, A^2_t\} \cup A_{k-1} \setminus \{A_t\}$ denotes the corresponding partition of $A$ which replaces $A_t \in A_{k-1}$ by $A^1_t$ and $A^2_t$.

If there exists a block of $A_k$ containing more than 2 objects, proceed to the next step. Otherwise, terminate the algorithm.

It is evident that the algorithm terminates in finite steps. Notice that the outcome of the algorithm varies according to the sequence of dichotomous refinements. Henceforth, we fix a sequence of dichotomous refinements, and suppose that the algorithm terminates at step $K$. At each step $1 \leq k \leq K$, by refining the domain from $\mathbb{D}(\Omega_{k-1})$ to $\mathbb{D}(\Omega_k)$, we reduce the number of instances exhibiting the local elevating property. At the termination step $K$, we observe that the partition $A_K$ contains $K + 1$ blocks, each block contains no more than 2 objects, and $\Omega_K$ collects $2^K$ restricted tier structures over the $K + 1$ blocks. Therefore, we obtain a union of $2^K$ restricted tier domains $\mathbb{D}(\Omega_K) = \bigcup_{\mathcal{P} \in \Omega_K} \mathbb{D}(\mathcal{P})$. Moreover, the next lemma shows that $\mathbb{D}(\Omega_K)$ violates both the local elevating and double elevating properties.

**Lemma 3** Domain $\mathbb{D}(\Omega_K)$ avoids both the local elevating and double elevating properties.

The proof of Lemma 3 is put in Appendix E.

More importantly, the following proposition shows that domain $\mathbb{D}(\Omega_K)$ admits an $sd$-$SP$, $sd$-$Eff$ and $sd$-$EF$ rule.

**Proposition 1** Domain $\mathbb{D}(\Omega_K)$ admits an $sd$-$SP$, $sd$-$Eff$ and $sd$-$EF$ rule.

The proof of Proposition 1 is put in Appendix F.

### 5 Conclusion

In this paper, we have shown that if a weakly connected domain admits an $sd$-$SP$, $sd$-$Eff$ and $ETE$ rule, it is a restricted tier domain, and this desirable rule is uniquely the PS rule. Normatively, we treat our domain characterization as a negative result, since a restricted tier structure almost requires that all agents have the same preference, and it implies the nonexistence of $sd$-$SP$, $sd$-$Eff$ and $ETE$ rules on almost all domains studied in the literature.

Our results help in understanding the boundary between possibilities and impossibilities on designing a desirable random assignment rule. Within weakly connected domains, the exact boundary is identified. Beyond weakly connected domains, although the exact boundary is not depicted, we present an algorithm to construct unions of restricted tier domains, where possibility holds.
Appendix

A Details related to Remark 1

In this appendix, we first introduce six restricted domains, and then show that the universal domain and all these six restricted domains are weakly connected domains. In particular, three restricted domains are defined in the one-dimensional setting: the single-peaked domain, the single-dipped domain and maximal single-crossing domains. The other three are defined in the multi-dimensional setting: the separable domain, the top-separable domain and the multidimensional single-peaked domain.

All three restricted domains in the one-dimensional setting share a common feature: Objects are exogenously arranged on a linear order <. For notational convenience, let a ⪯ b denote either a < b or a = b.

First, a preference $P_i$ is single-peaked on < if, for each pair of objects at the same side of the peak $r_1(P_i)$, the one closer to the peak in < is always preferred, i.e., $[b < a \leq r_1(P_i) \text{ or } r_1(P_i) < a < b] \Rightarrow [a P_i b]$. The single-peaked domain is the set containing all preferences single-peaked on <.

Second, a single-dipped preference performs exactly opposite to a single-peaked preference. In particular, a preference $P_i$ is single-dipped on < if $[b < a \leq r_{1\setminus i}(P_i) \text{ or } r_{1\setminus i}(P_i) < a \leq b] \Rightarrow [b P_i a]$. The single-dipped domain is the set containing all preferences single-dipped on <.

Third, to define a single-crossing domain, an exogenous linear order < needs to be fixed between preferences. A domain $\mathbb{D}$ is single-crossing on ($<$, $<$) if for all $a, b \in A$ with $a < b$ and $P_i, P'_i \in \mathbb{D}$ with $P_i \prec P'_i$, we have $[a P_i b] \Rightarrow [a P_i b]$ and $[b P_i a] \Rightarrow [b P_i a]$. Furthermore, a single-crossing domain is maximal if it has the maximal cardinality $\frac{|A|\times(|A|-1)}{2} + 1$.

According to Propositions 3 and 4 of Carroll (2012) and Propositions 4.1 and 4.2 of Sato (2013), we know that in the universal domain, the single-peaked domain and a maximal single-crossing domain, two distinct preferences are connected via a sequence of adjacent preferences. Therefore, they are all weakly connected domains. The single-dipped domain is also a weakly connected domain as the same argument for the single-peaked domain applies.

In the multidimensional setting, the object set $A$ is assumed to have a Cartesian product structure, i.e., $A = \times_{s \in M} A^s$ where (i) $M \equiv \{1, \ldots, m\}$ is finite and $m \geq 2$, and (ii) for each $s \in M$, the component set $A^s$ contains finite and at least two elements. Thus, an object can be represented by an $m$-tuple, i.e., $a \equiv (a^1, \ldots, a^m) \equiv (a^s, a^{-s})$.

First, a preference $P_i$ is separable if for each $s \in M$, a marginal preference over all elements of the component set $A^s$ can be independently elicited from $P_i$, i.e., for all $a^s, b^s \in A^s$, we have $[(a^s, x^{-s}) P_i (b^s, x^{-s}) \text{ for some } x^{-s} \in A^{-s}] \Rightarrow [(a^s, y^{-s}) P_i (b^s, y^{-s}) \text{ for all } y^{-s} \in A^{-s}]$. The separable domain is the set containing all separable preferences.

Second, a top-separable preference is less restricted than a separable preference. A preference $P_i$, say $r_1(P_i) = x \equiv (x^s)_{s \in M}$, is top-separable if for all $s \in M$ and $a^s, b^s \in A^s$, we have $[a^s = x^s \text{ and } b^s \neq x^s] \Rightarrow [(a^s, z^{-s}) P_i (b^s, z^{-s}) \text{ for all } z^{-s} \in A^{-s}]$. The top-separable domain is the set containing all top-separable preferences.

Last, to introduce the multidimensional single-peaked domain, an additional restriction must be imposed on the object set: For each $s \in M$, all element of $A^s$ are exogenously arranged on a linear order $<^s$. Symmetrically, let $a^s \leq b^s$ denote either $a^s < b^s$ or $a^s = b^s$. Thus, all objects...
are located on a product of linear orders, $x_{s \in M} <^s$. Given $x, y \in A$, let $MB(x, y) = \{a \in A : x^a \leq^s y^a\}$ denote the minimal box collecting all objects located between $x$ and $y$. Now, a preference $P_i$ is multidimensional single-peaked on $x_{s \in M} <^s$ if for all $a, b \in A$, we have $[a \in MB(r_i(P_i), b) \land a \neq b] \Rightarrow [a, b]$. The multidimensional single-peaked domain is the set containing all admissible preferences.

Chatterji and Zeng (2018) introduce a natural generalization of adjacency in the multidimensional setting: Preferences $P_i$ and $P'_i$ are adjacent$^+$ if the following two conditions are satisfied:

1. Both $P_i$ and $P'_i$ are separable preferences.
2. There exist $s \in M$ and $a^s, b^s \in A^s$ such that $(a^s, z^{-s}) P_i! (b^s, z^{-s})$ and $(b^s, z^{-s}) P'_i! (a^s, z^{-s})$ for all $z^{-s} \in A^{-s}$, and $[x P_i y] \Leftrightarrow [x P'_i y]$ for all $(x, y) \notin \{(a^s, z^{-s}), (b^s, z^{-s})\} : z^{-s} \in A^{-s}$.

One would note that across the adjacent$^+$ pair $P_i$ and $P'_i$, $|A^{-s}|$ pairs of objects are locally switched, while all other objects are commonly ranked. Therefore, the notion of adjacency$^+$ is a special case of our neighborhood. Chatterji and Zeng (2018) show that in each one of the three multidimensional domains, two distinct preferences are connected via a sequence of preferences such that each consecutive pair is either adjacent or adjacent$^+$. Therefore, all these three multidimensional domains are weakly connected domains.

## B Proof of Lemma 1

Suppose that $\mathbb{D}$ contains preferences $\tilde{P}_i, P_i, P'_i$ of Definition 3 or 4. For notational convenience, let $B \equiv B(\tilde{P}_i, a) = B(P_i, a) = B(\tilde{P}_i, b)$. Let $\varphi : \mathbb{D}^n \rightarrow \mathcal{L}$ be an sd-SP, sd-Eff and ETE rule. Let $\tilde{n} \equiv n$ if $n$ is even, and $\tilde{n} \equiv \frac{n-1}{2}$ if $n$ is odd. We search for a contradiction. We consider six groups of preference profiles: **Profile Groups I - VI** (Table 3 below). Note that every preference profile in Table 3 consists only of preference(s) of $\tilde{P}_i, P_i$ and $P'_i$.

### Table 3: Preference Profile Groups I - VI

<table>
<thead>
<tr>
<th>Profile Group I: $n$ is either even or odd</th>
<th>Profile Group II: $n$ is either even or odd</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P^{1.0} = (P_1, \ldots, P_n)$</td>
<td>$P^{2.0} = (\tilde{P}_1, \ldots, \tilde{P}_n)$</td>
</tr>
<tr>
<td>$P^{1.m} = (\tilde{P}_1, \ldots, \tilde{P}<em>m, P</em>{m+1}, \ldots, P_n)$,</td>
<td>$P^{2.m} = (\tilde{P}<em>1, \ldots, \tilde{P}</em>{n-m}, P_{n-m+1}, \ldots, P_n)$,</td>
</tr>
<tr>
<td>where $m = 1, \ldots, \tilde{n}$.</td>
<td>where $m = 1, \ldots, \tilde{n}$.</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Profile Group III: $n$ is either even or odd</td>
<td>Profile Group IV: $n$ is either even or odd</td>
</tr>
<tr>
<td>$P^{3.1} = (P_1, \ldots, P_{n-1}, P_n)$</td>
<td>$P^{4.1} = (\tilde{P}<em>1, \ldots, \tilde{P}</em>{n-2}, \tilde{P}_{n-1}, P_n)$</td>
</tr>
<tr>
<td>$P^{3.m} = (\tilde{P}<em>1, \ldots, \tilde{P}</em>{m-1}, P_m, \ldots, P_{n-1}, \tilde{P}_n)$,</td>
<td>$P^{4.m} = (\tilde{P}<em>1, \ldots, \tilde{P}</em>{n-m}, P_{n-m+1}, \ldots, P_{n-1}, \tilde{P}_n)$,</td>
</tr>
<tr>
<td>where $m = 2, \ldots, \tilde{n}, \tilde{n} + 1$.</td>
<td>where $m = 2, \ldots, \tilde{n}$.</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>Profile Group V: $n$ is odd</td>
<td>Profile Group VI: $n$ is odd</td>
</tr>
<tr>
<td>$P^{5.1} = (P_1, \ldots, P_{n-2}, \tilde{P}_{n-1}, P_n)$</td>
<td>$P^{6.1} = (\tilde{P}<em>1, \ldots, \tilde{P}</em>{n-2}, \tilde{P}_{n-1}, P_n)$</td>
</tr>
<tr>
<td>$P^{5.m} = (\tilde{P}<em>1, \ldots, \tilde{P}</em>{m-1}, P_m, \ldots, P_{n-2}, P_{n-1}, \tilde{P}_n)$,</td>
<td>$P^{6.m} = (\tilde{P}<em>1, \ldots, \tilde{P}</em>{n-m}, P_{n-m+1}, \ldots, P_{n-2}, \tilde{P}_{n-1}, P_n)$,</td>
</tr>
<tr>
<td>where $m = 2, \ldots, \tilde{n}, \tilde{n} + 1$.</td>
<td>where $m = 3, \ldots, \tilde{n}$.</td>
</tr>
</tbody>
</table>

**Claim 4** Under both the local elevating and double elevating properties, at every profile $\tilde{P}$ of profile groups I-VI, we have $\varphi_{iB}(\tilde{P}) = \frac{k-1}{n}$ for all $i \in I$. 

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Proof: The claim follows from a repeated application of sd-SP and ETE. The verification is routine, and we hence omit the detailed proof.

Claim 5 Under the local elevating property, at every profile $\tilde{P}$ of profile groups I-VI, we have $\sum_{x\in\{a,b,c\}} \varphi_{ix}(\tilde{P}) = \frac{3}{n}$ for all $i \in I$.

Proof: The claim follows from a repeated application of sd-SP and ETE. The verification is routine, and we hence omit the detailed proof.

Claim 6 In profile group I, for each $m = 0, 1, \ldots, \bar{n}$, at $P^{1,m} = (\hat{P}_1, \ldots, \hat{P}_m, P_{m+1}, \ldots, P_N)$, the random assignment $\varphi(P^{1,m})$ over $a, b, c$ and $d$ is specified below.

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, $\ldots$, $m$:</td>
<td>$0$</td>
<td>$\frac{2}{n}$</td>
<td>$\frac{1}{n}$</td>
</tr>
<tr>
<td>$m + 1, \ldots, n$:</td>
<td>$\frac{1}{n-m}$</td>
<td>$\frac{n-2m}{n(n-m)}$</td>
<td>$\frac{1}{n}$</td>
</tr>
</tbody>
</table>

Under the local elevating property

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, $\ldots$, $m$:</td>
<td>$0$</td>
<td>$\frac{2}{n}$</td>
<td>$\frac{1}{n}$</td>
</tr>
<tr>
<td>$m + 1, \ldots, n$:</td>
<td>$\frac{1}{n-m}$</td>
<td>$\frac{n-2m}{n(n-m)}$</td>
<td>$\frac{1}{n-m}$</td>
</tr>
</tbody>
</table>

Under the double elevating property

Proof: The proof consists of 2 steps. In the first step, we specify the random assignment over $a$ and $b$ at every preference profile under both the local elevating and double elevating properties. In the second step, by Claim 5, we first automatically obtain the random assignment over $c$ and $d$. Moreover, the particular profile group I and feasibility imply $\varphi_{ia}(P^{1,0}) = \frac{1}{n}$ and $\varphi_{ib}(P^{1,0}) = \frac{1}{n}$ for all $i \in I$. Next, we provide an induction hypothesis: Given $0 < m \leq \bar{n}$, for all $0 \leq l < m$, the random assignment $\varphi(P^{1,l})$ over $a$ and $b$ is specified below:

<table>
<thead>
<tr>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, $\ldots$, $m$:</td>
<td>$0$</td>
</tr>
<tr>
<td>$m + 1, \ldots, n$:</td>
<td>$\frac{1}{n-l}$</td>
</tr>
</tbody>
</table>

We specify the random assignment $\varphi(P^{1,m})$ over $a$ and $b$ to complete the verification of the induction hypothesis.

First, by the induction hypothesis, sd-SP implies $\varphi_{mb}(P^{1,m}) + \varphi_{ma}(P^{1,m}) = \varphi_{mb}(P^{1,m-1}) + \frac{2}{n}$. Then, ETE implies $\varphi_{ib}(P^{1,m}) + \varphi_{ia}(P^{1,m}) = \frac{2}{n}$ for all $i = 1, \ldots, m$. Furthermore, by ETE and feasibility, we have $\varphi_{jb}(P^{1,m}) + \varphi_{ja}(P^{1,m}) = \frac{2-m \times \frac{2}{n}}{n-m} = \frac{2}{n}$ for all $j = m + 1, \ldots, n$.

Next, we assert $\varphi_{ia}(P^{1,m}) = 0$ for all $i = 1, \ldots, m$. Suppose not, i.e., there exists $i^* \in \{1, \ldots, m\}$ such that $\varphi_{i^*a}(P^{1,m}) > 0$. Since every agent other than $1, \ldots, m$ prefers $a$ to $b$, sd-Eff implies $\varphi_{jb}(P^{1,m}) = 0$ for all $j = m + 1, \ldots, n$. Then, ETE and feasibility imply $\varphi_{i^*b}(P^{1,m}) = \frac{1}{m}$.

Consequently, $\frac{2}{n} = \varphi_{i^*a}(P^{1,m}) + \frac{1}{m}$. However, since $m \leq \bar{n}$, one can easily show $\frac{2}{n} \leq \frac{1}{m}$. Contradiction! Therefore, $\varphi_{ia}(P^{1,m}) = 0$ for all $i = 1, \ldots, m$. Thus, $\varphi_{ib}(P^{1,m}) = \frac{2}{n}$ for all $i = 1, \ldots, m$. Then, by ETE and feasibility, we have $\varphi_{ja}(P^{1,m}) = \frac{1}{n-m}$ and $\varphi_{jb}(P^{1,m}) = \frac{1-m \times \frac{2}{n}}{n-m} = \frac{n-2m}{n(n-m)}$ for all $j = m + 1, \ldots, n$. This completes the verification of the induction hypothesis and Step 1.
Step 2: Next, under the local elevating property, by Claim 5, we automatically obtain the random assignment over $c$. Under the double elevating property, similar to Step 1, from $P^{1,0}$ to $P^{1,n}$ step by step, by repeatedly applying $sd-SP$, $sd-Eff$, $ETE$ and feasibility, we have the random assignment over $c$ and $d$. This proves the claim.

Claim 7 In profile group II, for each $m = 0, 1, \ldots, \bar{n}$, at $P^{2,m} = (\hat{P}_1, \ldots, \hat{P}_{n-m}, P_{n-m+1}, \ldots, P_n)$, the random assignment $\varphi(P^{2,m})$ over $a$, $b$, $c$ and $d$ is specified below.

<table>
<thead>
<tr>
<th></th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1, \ldots, n-m$</td>
<td>$\frac{n-2m}{n(n-m)}$</td>
<td>$\frac{1}{n-m}$</td>
<td>$\frac{1}{n}$</td>
<td>$\frac{n-2m}{n(n-m)}$</td>
</tr>
<tr>
<td>$n-m+1, \ldots, n$</td>
<td>$\frac{2}{n}$</td>
<td>$0$</td>
<td>$\frac{1}{n}$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

Under the local elevating property

Proof: The verification is symmetric to the proof of Claim 6.

Claim 8 In profile group III, under both the local elevating and double elevating properties, at $P^{3,1} = (P_1, \ldots, P_{n-1}, \bar{P}_n)$, the random assignment $\varphi(P^{3,1})$ over $a$, $b$ and $c$ is specified below.

<table>
<thead>
<tr>
<th></th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1, \ldots, n-1$</td>
<td>$\frac{1}{n}$</td>
<td>$\frac{1}{n-1}$</td>
<td>$\frac{n-2}{n(n-1)}$</td>
</tr>
<tr>
<td>$n$</td>
<td>$\frac{1}{n}$</td>
<td>$0$</td>
<td>$\frac{2}{n}$</td>
</tr>
</tbody>
</table>

Proof: The proof consists of 2 steps.

Step 1: First, by Claim 6, $sd-SP$ implies $\varphi_{na}(P^{3,1}) = \varphi_{na}(P^{1,0}) = \frac{1}{n}$. Next, $sd-Eff$ implies $\varphi_{nb}(P^{3,1}) = 0$. Then, by $ETE$ and feasibility, we have $\varphi_{ia}(P^{3,1}) = \frac{1}{n}$ and $\varphi_{ib}(P^{3,1}) = \frac{1}{n-1}$ for all $i = 1, \ldots, n-1$.

Step 2: First, by Claim 6, $sd-SP$ implies $\varphi_{nc}(P^{3,1}) + \varphi_{nb}(P^{3,1}) = \varphi_{nb}(P^{1,0}) + \varphi_{nc}(P^{1,0}) = \frac{2}{n}$. Since $\varphi_{nb}(P^{3,1}) = 0$ in Step 1, we have $\varphi_{nc}(P^{3,1}) = \frac{2}{n}$. Then, by $ETE$ and feasibility, we have $\varphi_{ic}(P^{3,1}) = \frac{1-\frac{2}{n-1}}{n-1} = \frac{n-2}{n(n-1)}$ for all $i = 1, \ldots, n-1$.

Claim 9 In profile group III, for each $m = 2, \ldots, \bar{n}$ (if $n$ is even), or $m = 2, \ldots, \bar{n}, \bar{n} + 1$ (if $n$ is odd), at $P^{3,m} = (\hat{P}_1, \ldots, \hat{P}_{n-m}, P_{n-m+1}, \ldots, \bar{P}_n)$, the random assignment $\varphi(P^{3,m})$ over $a$, $b$ and $c$ is specified below.

<table>
<thead>
<tr>
<th></th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1, \ldots, m-1$</td>
<td>$0$</td>
<td>$\alpha(m)$</td>
<td>$\frac{3}{n} - \alpha(m)$</td>
</tr>
<tr>
<td>$m, \ldots, n-1$</td>
<td>$\frac{1}{n-(m-1)}$</td>
<td>$\frac{3}{n-\alpha(m)}$</td>
<td>$\frac{3}{n} - \frac{1}{n-\alpha(m)}$</td>
</tr>
<tr>
<td>$n$</td>
<td>$\frac{1}{n-(m-1)}$</td>
<td>$0$</td>
<td>$\frac{3}{n} - \frac{1}{n-\alpha(m)}$</td>
</tr>
</tbody>
</table>

Under the local elevating property

<table>
<thead>
<tr>
<th></th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1, \ldots, m-1$</td>
<td>$0$</td>
<td>$\alpha(m)$</td>
<td>$0$</td>
</tr>
<tr>
<td>$m, \ldots, n-1$</td>
<td>$\frac{1}{n-(m-1)}$</td>
<td>$\frac{3}{n-\alpha(m)}$</td>
<td>$\frac{n-2}{n(n-m)}$</td>
</tr>
<tr>
<td>$n$</td>
<td>$\frac{1}{n-(m-1)}$</td>
<td>$0$</td>
<td>$\frac{2}{n}$</td>
</tr>
</tbody>
</table>

Under the double elevating property
where \( \alpha(m) = \frac{2n^2 - (2m - 1)n + 1}{n(n-1)[n-(m-1)]} \).

Proof: The proof consists of 3 steps. In the first step, we specify the random assignment over \( a \) and \( b \) under both the local elevating and double elevating properties. By Claim 5, we then automatically obtain the random assignment over \( c \) under the local elevating property. In the second step, we make two observations on the random assignment under the double elevating property, while the last step specifies the random assignment over \( c \) under the double elevating property.

Step 1: By Claim 6, sd-SP implies \( \varphi_{na}(P^{3,2}) = \varphi_{na}(P^{1,1}) = \frac{1}{n-1} \). Symmetrically, by Claim 8, sd-SP implies \( \varphi_{lb}(P^{3,2}) + \varphi_{la}(P^{3,2}) = \varphi_{lb}(P^{3,1}) + \varphi_{la}(P^{3,1}) = \frac{1}{n} + \frac{1}{n-1} \). Next, by sd-Eff, we have \( \varphi_{nb}(P^{3,2}) = 0 \) and \( \varphi_{la}(P^{3,2}) = 0 \). Thus, \( \varphi_{lb}(P^{3,2}) = \frac{1}{n} + \frac{1}{n-1} = \frac{2n-1}{n(n-1)} = \alpha(2) \). Last, by ETE and feasibility, we have \( \varphi_{ia}(P^{3,2}) = \frac{1}{n-1} \) and \( \varphi_{ib}(P^{3,2}) = \frac{1}{n-2} \) for all \( i = 2, \ldots, n-1 \).

Next, we adopt an induction hypothesis: Given \( 2 < m \leq \bar{n} \) (if \( \bar{n} = \frac{n}{2} \)), or \( 2 < m \leq \bar{n} + 1 \) (if \( \bar{n} = \frac{n-1}{2} \)), for all \( 2 \leq l < m \), we have the random assignment \( \varphi(P^{3,m}) \) over \( a \) and \( b \) as follows:

\[
\begin{array}{ccc}
  a & b \\
 1, \ldots, l-1: & 0 & \alpha(l) \\
l, \ldots, n-1: & \frac{1}{n-(l-1)} & \frac{1-(l-1)\times\alpha(l)}{n-l} \\
n: & \frac{1}{n-(l-1)} & 0
\end{array}
\]

We specify the random assignment \( \varphi(P^{3,m}) \) over \( a \) and \( b \) to complete the verification of the induction hypothesis.

By Claim 6, sd-SP implies \( \varphi_{na}(P^{3,m}) = \varphi_{na}(P^{1,m-1}) = \frac{1}{n-(m-1)} \). By sd-Eff, we have \( \varphi_{nb}(P^{3,m}) = 0 \). Next, by sd-SP and the induction hypothesis, we have

\[
\begin{align*}
\varphi_{m-1b}(P^{3,m}) + \varphi_{m-1a}(P^{3,m}) &= \varphi_{m-1a}(P^{3,m-1}) + \varphi_{m-1b}(P^{3,m-1}) \\
&= \frac{1}{n-(m-2)} + \frac{1-(m-2) \times \alpha(m-1)}{n-(m-1)} \\
&= \frac{1}{n-(m-2)} + \frac{1-(m-2) \times \left[\frac{2n^2 - (2m-1)n + 1}{n(n-1)[n-(m-2)]}\right]}{n-(m-1)} \\
&= \frac{2n^2 - (2m-1)n + 1}{n(n-1)[n-(m-1)]} = \alpha(m).
\end{align*}
\]

Furthermore, ETE implies \( \varphi_{ib}(P^{3,m}) + \varphi_{ia}(P^{3,m}) = \alpha(m) \) for all \( i = 1, \ldots, m-1 \).

Last, we show \( \varphi_{ia}(P^{3,m}) = 0 \) for all \( i = 1, \ldots, m-1 \). Suppose not, i.e., there exists \( i^* \in \{1, \ldots, m-1\} \) such that \( \varphi_{i^*a}(P^{3,m}) > 0 \). Since every agent other than \( 1, \ldots, m-1 \) prefers \( a \) to \( b \), sd-Eff implies \( \varphi_{jb}(P^{3,m}) = 0 \) for all \( j = m, \ldots, n \). Then, ETE and feasibility imply \( \varphi_{i^*b}(P^{3,m}) = \frac{1}{m-1} \). Consequently, \( \alpha(m) = \varphi_{i^*a}(P^{3,m}) + \varphi_{i^*b}(P^{3,m}) > \frac{1}{m-1} \). However, one can easily show

\[
\frac{1}{m-1} - \alpha(m) = \frac{1}{m-1} - \frac{2n^2 - (2m-1)n + 1}{n(n-1)[n-(m-1)]} = \frac{n(n-m)[n-2(m-1)] - (m-1)}{(m-1)n(n-1)[n-(m-1)]} \geq \left\{ \begin{array}{ll}
\frac{2n^2-n+2}{2(n-1)n(n-m-1)} & > 0 \text{ if } n \text{ is even}, \\
\frac{(n-1)^2}{2(n-1)n(n-m-1)} & > 0 \text{ if } n \text{ is odd}.
\end{array} \right.
\]

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Contradiction! Therefore, \( \varphi_{ia}(P^{3,m}) = 0 \) for all \( i = 1, \ldots, m - 1 \). Hence, \( \varphi_{ia}(P^{3,m}) = \alpha(m) \) for all \( i = 1, \ldots, m - 1 \). Last, by ETE and feasibility, we have \( \varphi_{ja}(P^{3,m}) = \frac{1 - \frac{1}{n}(m-1)}{n-m} = \frac{1}{n-m} \) and \( \varphi_{jb}(P^{3,m}) = \frac{1-(m-1)\alpha(m)}{n-m} \) for all \( j = m, \ldots, n - 1 \). In conclusion, we have the random assignment \( \varphi(P^{3,m}) \) over \( a \) and \( b \) as follows:

\[
\begin{array}{ccc}
a & b \\
1, \ldots, m - 1: & 0 & \alpha(m) \\
m, \ldots, n - 1: & \frac{1}{n-(m-1)} & \frac{1-(m-1)\alpha(m)}{n-m} \\
\end{array}
\]

This completes the verification of the induction hypothesis and Step 1.

**Step 2:** Under the local elevating property, by Claim 5, we have the random assignment over \( c \) at each preference profile. We focus on specifying the random assignment over \( c \) under the double elevating property. We first make two observations: For each \( m = 2, \ldots, \tilde{n} \) (if \( n \) is even), or \( m = 2, \ldots, \tilde{n}, \tilde{n} + 1 \) (if \( n \) is odd), at \( P^{3,m} = (P_1, \ldots, P_{m-1}, P_m, \ldots, P_{n-1}, P_n) \).

**Observation 1.** if \( r_{k+3}(\tilde{P}_1) = d \), we have \( \sum_{x \in \{a,b,c,d\}} \varphi_{ix}(P^{3,m}) = \frac{d}{n} \) for all \( i \in I \).

**Observation 2.** if \( r_{k+3}(\tilde{P}_1) \neq d \), we have \( \varphi_{ic}(P^{3,m}) + \varphi_{id}(P^{3,m}) = \frac{2}{n} \) for all \( i \in I \).

The first observation is similar to Claim 5. We prove the second observation. Let \( r_{k+3}(\tilde{P}_1) \neq d \). Consequently, \( \varphi_{nc}(P^{3,2}) = 0 \) by sd-Eff. Next, by Claim 6, sd-SP implies \( \varphi_{nc}(P^{3,2}) + \varphi_{nb}(P^{3,2}) = \varphi_{nb}(P^{1,1}) + \varphi_{nc}(P^{1,1}) = \frac{2}{n} \). Since \( \varphi_{nb}(P^{3,2}) = 0 \) in Step 1, we have \( \varphi_{nc}(P^{3,2}) = \frac{2}{n} \) and hence \( \varphi_{nc}(P^{3,2}) + \varphi_{nd}(P^{3,2}) = \frac{2}{n} \). Next, recall profile \( P^{3,1} = (P_1, \ldots, P_{n-1}, P_n) \). Since \( r_{k+3}(\tilde{P}_1) \neq d \), sd-Eff implies \( \varphi_{nd}(P^{3,1}) = 0 \). Then, ETE and feasibility imply \( \varphi_{id}(P^{3,1}) = \frac{1}{n-1} \). Hence, by Claim 8, we have \( \varphi_{1c}(P^{3,1}) + \varphi_{id}(P^{3,1}) = \frac{n-2}{n(n-1)} + \frac{1}{n-1} = \frac{2}{n} \). Then, sd-SP implies \( \varphi_{ic}(P^{3,2}) + \varphi_{1c}(P^{3,2}) = \varphi_{1c}(P^{3,1}) + \varphi_{id}(P^{3,1}) = \frac{2}{n} \). Last, by ETE and feasibility, we have \( \varphi_{ic}(P^{3,2}) + \varphi_{id}(P^{3,2}) = \frac{2-\frac{2}{n-1}}{n-1} = \frac{2}{n} \) for all \( i = 2, \ldots, n - 1 \).

Next, we adopt an induction hypothesis: Given \( 2 < m \leq \tilde{n} \) (if \( n \) is even), or \( 2 < m \leq \tilde{n} + 1 \) (if \( n \) is odd), for all \( 2 \leq l < m \), we have \( \varphi_{ic}(P^{3,l}) + \varphi_{id}(P^{3,l}) = \frac{2}{n} \) for all \( i \in I \). We show \( \varphi_{ic}(P^{3,m}) + \varphi_{id}(P^{3,m}) = \frac{2}{n} \) for all \( i \in I \).

First, by sd-Eff, we have \( \varphi_{ncd}(P^{3,m}) = 0 \). Next, by Claim 6, sd-SP implies \( \varphi_{nc}(P^{3,m}) + \varphi_{nb}(P^{3,m}) + \varphi_{nc}(P^{1,m-1}) + \varphi_{nc}(P^{1,m-1}) = \frac{2}{n} \). Since \( \varphi_{nb}(P^{3,m}) = 0 \) in Step 1, we have \( \varphi_{nc}(P^{3,m}) = \frac{2}{n} \), and hence \( \varphi_{nc}(P^{3,m}) + \varphi_{id}(P^{3,m}) = \frac{2}{n} \). Next, by sd-SP and the induction hypothesis, we have \( \varphi_{m-1c}(P^{3,m}) + \varphi_{m-1d}(P^{3,m}) = \varphi_{m-1c}(P^{3,m-1}) + \varphi_{m-1d}(P^{3,m-1}) = \frac{2}{n} \). Then, ETE implies \( \varphi_{ic}(P^{3,m}) + \varphi_{id}(P^{3,m}) = \frac{2}{n} \) for all \( i = 1, \ldots, m - 1 \). Last, by ETE and feasibility, we have \( \varphi_{ic}(P^{3,m}) + \varphi_{id}(P^{3,m}) = \frac{2-\frac{2}{n-1}}{n-1} = \frac{2}{n} \) for all \( j = m, \ldots, n - 1 \). This completes the verification of the induction hypothesis, and hence proves the second observation.

**Step 3:** Now, we show the random assignment over \( c \) at each preference profile.

First, by Claim 6, sd-SP implies \( \varphi_{nc}(P^{3,2}) + \varphi_{nb}(P^{3,2}) = \varphi_{nb}(P^{1,1}) + \varphi_{nc}(P^{1,1}) = \frac{2}{n} \). Since \( \varphi_{nb}(P^{3,2}) = 0 \) in Step 1, we have \( \varphi_{nc}(P^{3,2}) = \frac{2}{n} \). Next, sd-Eff implies \( \varphi_{ic}(P^{3,2}) = 0 \). Then, by ETE and feasibility, we have \( \varphi_{ic}(P^{3,2}) = \frac{1-\frac{2}{n}}{n-2} = \frac{n-2}{n(n-2)} \) for all \( i = 2, \ldots, n - 1 \).

Next, we adopt an induction hypothesis: Given \( 2 < m \leq \tilde{n} \) (if \( n \) is even), or \( 2 < m \leq \tilde{n} + 1 \) (if \( n \) is odd), for all \( 2 \leq l < m \), we have (i) \( \varphi_{ic}(P^{3,l}) = 0 \) for all \( i = 1, \ldots, l-1 \), (ii) \( \varphi_{ic}(P^{3,l}) = \frac{n-2}{n(n-2)} \) for all \( i \in I \).
for all \( j = 1, \ldots, n - 1 \), and (iii) \( \varphi_{nc}(P^{3,j}) = \frac{2}{n} \). We show \( \varphi_{ic}(P^{3,m}) = 0 \) for all \( i = 1, \ldots, m - 1 \), \( \varphi_{jc}(P^{3,m}) = \frac{n-1}{n(n-m)} \) for all \( j = m, \ldots, n - 1 \), and \( \varphi_{nc}(P^{3,m}) = \frac{2}{n} \).

First, by Claim 6, \( sd-SP \) implies \( \varphi_{nc}(P^{3,m}) + \varphi_{nb}(P^{3,m}) = \varphi_{nb}(P^{1,m-1}) + \varphi_{nc}(P^{1,m-1}) = \frac{2}{n} \).

Since \( \varphi_{nb}(P^{3,m}) = 0 \) in Step 1, we have \( \varphi_{nc}(P^{3,m}) = \frac{2}{n} \). Next, suppose that there exists \( i^* \in \{1, \ldots, m - 1\} \) such that \( \varphi_{ic}(P^{3,m}) > 0 \). Since every agent other than \( 1, \ldots, m - 1 \) prefers \( c \) to \( d \), \( sd-Eff \) implies \( \varphi_{jd}(P^{3,m}) = 0 \) for all \( j = m, \ldots, n \). Then, \( ETE \) and feasibility imply \( \varphi_{m-1d}(P^{3,m}) \geq \frac{1}{m-1} \). Hence, \( \varphi_{m-1d}(P^{3,m}) + \varphi_{m-1c}(P^{3,m}) > \frac{1}{m-1} \). If \( r_{k+3}(\hat{P}) = d \), by Observation 1, we have

\[
\frac{4}{n} = \left[ \varphi_{m-1d}(P^{3,m}) + \varphi_{m-1}(P^{3,m}) \right] + \left[ \varphi_{m-1d}(P^{3,m}) + \varphi_{m-1c}(P^{3,m}) \right] > \alpha(m) + \frac{1}{m-1}.
\]

However, if \( n \) is even, one can easily show that

\[
\left[ \alpha(m) + \frac{1}{m-1} \right] - \frac{4}{n} = \frac{2n^2 - (2m-1)n + 1}{n(n-1)[n-1]} + \frac{1}{m-1} - \frac{4}{n} \\
\geq \frac{2n^2 - (2m-1)n + 1}{n(n-1)[n-1]} + \frac{2}{n-2} - \frac{4}{n} \\
= \frac{5n^2 - 6mn + n + 8m - 10}{n(n-1)(n-2)[n-1]} \\
\geq \frac{2n^2 + n + 8m - 10}{n(n-1)(n-2)[n-1]} > 0,
\]

and if \( n \) is odd, one can easily show that

\[
\left[ \alpha(m) + \frac{1}{m-1} \right] - \frac{4}{n} = \frac{2n^2 - (2m-1)n + 1}{n(n-1)[n-1]} + \frac{1}{m-1} - \frac{4}{n} \\
\geq \frac{2n^2 - (2m-1)n + 1}{n(n-1)[n-1]} + \frac{2}{n-1} - \frac{4}{n} \\
= \frac{3n^2 + 5 - 4m}{n(n-1)[n-1]} \\
\geq \frac{n + 3}{n(n-1)[n-1]} > 0.
\]

Contradiction! If \( r_{k+3}(\hat{P}) \neq d \), by Observation 2, we have \( \frac{2}{n} = \varphi_{m-1d}(P^{3,m}) + \varphi_{m-1c}(P^{3,m}) > \frac{1}{m-1} \). However, one can easily observe \( \frac{2}{n} < \frac{1}{m-1} \). Contradiction! Therefore, \( \varphi_{ic}(P^{3,m}) = 0 \) for all \( i = 1, \ldots, m - 1 \). Last, by \( ETE \) and feasibility, we have \( \varphi_{jc}(P^{3,m}) = \frac{1-\frac{2}{n}}{n-1} = \frac{n-2}{n(n-m)} \) for all \( j = m, \ldots, n - 1 \). This completes the verification of the induction hypothesis. We hence finish the specification of the random assignment over \( c \) at each preference profile of profile group III under the double elevating property. This proves the claim.

\[\text{Claim 10} \text{ In profile group IV, for each } m = 1, \ldots, \bar{n}, \text{ at } P^{4,m} = (\hat{P}_1, \ldots, \hat{P}_{n-m}, P_{n-m+1}, \ldots, P_{n-1}, \bar{P}), \text{ the random assignment } \varphi(P^{4,m}) \text{ over } a, b \text{ and } c \text{ is specified below.}\]

\[
\begin{array}{c|ccc|c|ccc|c|}
 & a & b & c & & a & b & c \\
1, \ldots, n-m: & \frac{n-2m}{n(n-m)} & \frac{1}{n-m} & \frac{1}{n} & & 1, \ldots, n-m: & \frac{n-2m}{n(n-m)} & \frac{1}{n-m} & \frac{n-2m}{n(n-m)} \\
n-m+1, \ldots, n-1: & \frac{2}{n} & 0 & \frac{1}{n} & & n-m+1, \ldots, n-1: & \frac{2}{n} & 0 & \frac{2}{n} \\
n: & \frac{2}{n} & 0 & \frac{1}{n} & & n: & \frac{2}{n} & 0 & \frac{2}{n}
\end{array}
\]

Under the local elevating property Under the double elevating property
Proof: The proof of this claim consists of two steps. In the first step, we specify the random assignment over \(a\) and \(b\) under both the local elevating and the double elevating properties. By Claim 5, we then automatically obtain the random assignment over \(c\) under the local elevating property. In the second step, we specify the random assignment over \(c\) under the double elevating property.

**Step 1:** First, by Claim 7, \(sd-SP\) implies \(\varphi_{na}(P^{4,1}) = \varphi_{na}(P^{2,1}) = \frac{2}{n}\). Next, \(sd-Eff\) implies \(\varphi_{nb}(P^{4,1}) = 0\). Then, by \(ETE\) and feasibility, we have \(\varphi_{ib}(P^{4,1}) = \frac{1}{n-1}\) and \(\varphi_{id}(P^{4,1}) = \frac{1-\frac{2}{n}}{n-1} = \frac{n-2}{n(n-1)}\) for all \(i = 1, \ldots, n-1\).

Next, we provide an induction hypothesis: Given \(1 < m \leq \bar{n}\), for all \(2 \leq l < m\), we have the random assignment \(\varphi(P^{4,l})\) over \(a\) and \(b\) specified below.

\[
\begin{array}{c|c|c}
1, \ldots, n - l: & a & b \\
n - l + 1, \ldots, n - 1: & \frac{2}{n} & 0 \\
n: & \frac{2}{n} & 0 \\
\end{array}
\]

We specify the random assignment \(\varphi(P^{4,m})\) over \(a\) and \(b\) to complete the verification of the induction hypothesis.

First, by Claim 7, \(sd-SP\) implies \(\varphi_{na}(P^{4,m}) = \varphi_{na}(P^{2,m}) = \frac{2}{n}\). Next, \(sd-Eff\) implies \(\varphi_{nb}(P^{4,m}) = 0\). By the induction hypothesis, \(sd-SP\) implies \(\varphi_{n-m+1,a}(P^{4,m}) + \varphi_{n-m+1,b}(P^{4,m}) = \varphi_{n-m+1,b}(P^{4,m-1}) + \varphi_{n-m+1,a}(P^{4,m-1}) = \frac{2}{n}\). Then, \(ETE\) implies \(\varphi_{ja}(P^{4,m}) + \varphi_{jb}(P^{4,m}) = \frac{2}{n}\) for all \(j = n - m + 1, \ldots, n - 1\). Suppose that there exists \(j^* \in \{n - m + 1, \ldots, n - 1\}\) such that \(\varphi_{j^*b}(P^{4,m}) > 0\). Since every agent other than \(n - m + 1, \ldots, n - 1\) prefers \(b\) to \(a\), \(sd-Eff\) implies \(\varphi_{ja}(P^{4,m}) = 0\) for all \(i = 1, \ldots, n - m\). Consequently, \(ETE\) and feasibility imply \(\varphi_{ja}(P^{4,m}) = \frac{1-\frac{2}{m-1}}{m-1} = \frac{n-2}{n(m-1)}\).

Thus, one can easily show \(\frac{2}{n} \leq \frac{n-2}{n(m-1)}\). Therefore, \(\varphi_{j^*b}(P^{4,m}) = 0\) for all \(j = n - m + 1, \ldots, n - 1\). Hence, \(\varphi_{ja}(P^{4,m}) = \frac{2}{n}\) for all \(j = n - m + 1, \ldots, n - 1\). Last, by \(ETE\) and feasibility, we have \(\varphi_{ib}(P^{4,m}) = \frac{1}{n-m}\) and \(\varphi_{ia}(P^{4,m}) = \frac{1-\frac{2}{n-m}}{n-m} = \frac{n-2m}{n(n-m)}\) for all \(i = 1, \ldots, n - m\). In conclusion, we have the random assignment \(\varphi(P^{4,m})\) over \(a\) and \(b\) specified below.

\[
\begin{array}{c|c|c}
1, \ldots, n - m: & a & b \\
n - m + 1, \ldots, n - 1: & \frac{2}{n} & 0 \\
n: & \frac{2}{n} & 0 \\
\end{array}
\]

This completes the verification of the induction hypothesis and Step 1.

**Step 2:** Under the local elevating property, by applying Claim 5, we have the random assignment over \(c\) at each preference profile. We focus on specifying the random assignment over \(c\) at each preference profile under the double elevating property.

First, by Claim 7, \(sd-SP\) implies \(\varphi_{nc}(P^{4,1}) + \varphi_{nb}(P^{4,1}) = \varphi_{nc}(P^{2,1}) + \varphi_{nb}(P^{2,1}) = \frac{2}{n}\). Since \(\varphi_{nb}(P^{4,1}) = 0\) in Step 1, we have \(\varphi_{nc}(P^{4,1}) = \frac{2}{n}\). Then, by \(ETE\) and feasibility, we have \(\varphi_{ic}(P^{4,1}) = \frac{1-\frac{2}{n}}{n-1} = \frac{n-2}{n(n-1)}\). Meanwhile, note that \(sd-Eff\) implies \(\varphi_{nd}(P^{4,1}) = 0\). Therefore, \(\varphi_{id}(P^{4,1}) = \frac{1}{n-1}\) for all \(i = 1, \ldots, n - 1\) by \(ETE\) and feasibility. Hence, \(\varphi_{ic}(P^{4,1}) + \varphi_{id}(P^{4,1}) = \frac{2}{n}\) for all \(i \in I\).

Next, we adopt an induction hypothesis: Given \(1 < m \leq \bar{n}\), for all \(1 \leq l < m\), we have
(i) $\varphi_{ic}(P^{4,l}) = \frac{n-2l}{n(n-1)}$ for all $i = 1, \ldots, n - l,$
$\varphi_{jc}(P^{4,l}) = \frac{2}{n}$ for all $j = n - l + 1, \ldots, n - 1,$ and
$\varphi_{nc}(P^{4,l}) = \frac{2}{n},$ and

(ii) $\varphi_{ic}(P^{4,l}) + \varphi_{id}(P^{4,l}) = \frac{2}{n}$ for all $i \in I.$

We show that (i) $\varphi_{ic}(P^{4,m}) = \frac{n-2m}{n(n-m)}$ for all $i = 1, \ldots, n - m,$ $\varphi_{jc}(P^{4,m}) = \frac{2}{n}$ for all $j = n - m + 1, \ldots, n - 1,$ and $\varphi_{nc}(P^{4,m}) = \frac{2}{n},$ and (ii) $\varphi_{ic}(P^{4,m}) + \varphi_{id}(P^{4,m}) = \frac{2}{n}$ for all $i \in I.$

First, by Claim 7, sd-SP implies $\varphi_{nc}(P^{4,m}) + \varphi_{nb}(P^{4,m}) = \varphi_{nc}(P^{2,m}) + \varphi_{nb}(P^{2,m}) = \frac{2}{n}.$ Since $\varphi_{nb}(P^{4,m}) = 0$ in Step 1, we have $\varphi_{nc}(P^{4,m}) = \frac{2}{n}.$ Note that $\varphi_{nd}(P^{2,m}) = 0$ by Claim 7, $P_i$ and $\bar{P}_i$ have the same set of top $k + 2$ ranked objects, $d = r_{k+3}(P_i)$ and $d = r_{\nu}(\bar{P}_i)$ for some $\nu \geq k + 3.$ Therefore, sd-SP implies $0 = \varphi_{nd}(P^{2,m}) \geq \varphi_{nd}(P^{4,m}).$ Hence, $\varphi_{nd}(P^{4,m}) = 0.$ Thus, we have $\varphi_{ic}(P^{4,m}) + \varphi_{nd}(P^{4,m}) = \frac{2}{n}.$

Next, by sd-SP and the induction hypothesis, we have $\varphi_{n-m+1c}(P^{4,m}) + \varphi_{n-m+1d}(P^{4,m}) = \varphi_{n-m+1d}(P^{4,m-1}) + \varphi_{n-m+1c}(P^{4,m-1}) = \frac{2}{n}.$ Then, ETE implies $\varphi_{jc}(P^{4,m}) + \varphi_{jd}(P^{4,m}) = \frac{2}{n}$ for all $j = n - m + 1, \ldots, n - 1.$ Then, by ETE and feasibility, we have $\varphi_{ic}(P^{4,m}) + \varphi_{id}(P^{4,m}) = \frac{2-(m-1)\frac{2}{n}}{n-m} = \frac{2}{n}$ for all $i = 1, \ldots, n - m.$ Hence, $\varphi_{ic}(P^{4,m}) + \varphi_{id}(P^{4,m}) = \frac{2}{n}$ for all $i \in I.$

Next, suppose that there exists $j^* \in \{n - m + 1, \ldots, n - 1\}$ such that $\varphi_{j^*d}(P^{4,m}) > 0.$ Since every agent other than $n - m + 1, \ldots, n - 1$ prefers $d$ to $c,$ sd-Eff implies $\varphi_{ic}(P^{4,m}) = 0$ for all $i = 1, \ldots, n - m.$ Then, ETE and feasibility imply $\varphi_{j^*c}(P^{4,m}) = \frac{1-\frac{2}{n}}{m-1} = \frac{n-2}{n(n-1)}.$

Thus, $\frac{2}{n} = \varphi_{j^*c}(P^{4,m}) + \varphi_{j^*d}(P^{4,m}) \geq \frac{n-2}{n(n-1)}.$ However, one can easily show $\frac{2}{n} \leq \frac{n-2}{n(n-1)}.$ Contradiction! Therefore, $\varphi_{jd}(P^{4,m}) = 0$ for all $j = n - m + 1, \ldots, n - 1.$ Hence, $\varphi_{ic}(P^{4,m}) = \frac{2}{n}$ for all $j = n - m + 1, \ldots, n - 1.$

Last, by ETE and feasibility, we have $\varphi_{ic}(P^{4,m}) = \frac{1-(m-1)\frac{2}{n}}{n-m} = \frac{n-2m}{n(n-1)}$ for all $i = 1, \ldots, n - m.$ This completes the verification of the induction hypothesis. We hence finish the specification of the random assignment over $c$ at each preference profile under the double elevating property. This proves the claim.

Now, we induce the contradiction for the case of an even number of agents. Let $n \geq 4$ be an even integer. Thus, $\bar{n} = \frac{n}{2}.$ Notice that $P^{3,\bar{n}}$ and $P^{4,\bar{n}}$ differ exactly in agent $\bar{n}$’s preference, i.e., $P^\bar{n}_\bar{n} = P_{\bar{n}}$ and $P^\bar{n}_{\bar{n}} = \bar{P}_{\bar{n}}$ in Definition 3 or 4. Then, sd-SP implies $\varphi_{\bar{n}a}(P^{3,\bar{n}}) + \varphi_{\bar{n}b}(P^{3,\bar{n}}) = \varphi_{\bar{n}a}(P^{4,\bar{n}}) + \varphi_{\bar{n}b}(P^{4,\bar{n}}).$ Thus, by Claims 9 and 10, we have

$$0 = \left[ \varphi_{\bar{n}a}(P^{3,\bar{n}}) + \varphi_{\bar{n}b}(P^{3,\bar{n}}) \right] - \left[ \varphi_{\bar{n}a}(P^{4,\bar{n}}) + \varphi_{\bar{n}b}(P^{4,\bar{n}}) \right]$$

$$= \left[ \frac{1}{n} - \frac{1}{n-\frac{2}{n}} \right] + \frac{1-(\frac{n}{2}-1)\frac{2}{n}}{n-\frac{n}{2}} - \frac{2}{n} = \frac{2}{n^2(n-1)}.$$

Contradiction!

In conclusion, in the case of an even number of agents, domain $\mathbb{D}$ satisfying the local elevating property or the double elevating property admits no sd-SP, sd-Eff and ETE rule.

Before turning to the case of an odd number of agents, we make one note on the case of an even number of agents. In all Claims 6 - 10, only the specification of the random assignment over $a$ and $b$ (correspondingly, Step 1 in the proof of each claim) is used to establish the impossibility result. The specification of the random assignment over $c$ is established for the following-up investigation in the case of an odd number of agents.
Now, we consider the case of an odd number of agents. Henceforth, let \( n \geq 5 \) be an odd integer. Thus, \( \bar{n} = \frac{n-1}{2} \). Claim 11 below is a preparation which will be used in establishing the following-up claims.

**Claim 11** In profile group III, under both the local elevating and double elevating properties, for each \( m = 2, \ldots, \bar{n}, \bar{n} + 1 \), at \( P^{3,m} = (\bar{P}_1, \ldots, \bar{P}_{m-1}, P_m, \ldots, P_{n-1}, \bar{P}_n) \), we have \( \varphi_{n-1b}(P^{3,m}) + \varphi_{n-1c}(P^{3,m}) < \frac{2}{n-1} \).

**Proof:** We first consider the situation under the local elevating property. Fix \( 2 \leq m \leq \bar{n} + 1 \). By Claim 9, we know \( \varphi_{n-1b}(P^{3,m}) + \varphi_{n-1c}(P^{3,m}) = \frac{3}{n} - \frac{1}{n-(m-1)} \). Then, it is easy to show that
\[
\frac{3}{n} - \frac{1}{n-(m-1)} - \frac{2}{n-1} \leq \frac{3}{n} - \frac{1}{n-1} - \frac{2}{n-1} < 0.
\]

Next, we consider the situation under the double elevating property. By Claim 9, for each \( m = 2, \ldots, \bar{n}, \bar{n} + 1 \), we have
\[
\frac{2}{n-1} - \left[ \varphi_{n-1b}(P^{3,m}) + \varphi_{n-1c}(P^{3,m}) \right] = \frac{2}{n-1} - \left[ \frac{2}{n-1} - \frac{1}{n-1} \times \alpha(m) + \frac{n-2}{n(n-m)} \right] = \frac{2}{n-1} - \frac{2(n-1)^2(n-m+1)}{n(n-1)(n-m)(n-m+1)} < 0.
\]

This proves the claim.

**Claim 12** In profile group V, under both the local elevating and double elevating properties, at \( P^{5,1} = (P_1, \ldots, P_{n-2}, \bar{P}_{n-1}, P_n) \), the random assignment \( \varphi(P^{5,1}) \) over \( a, b \) and \( c \) is specified below.

\[
a = b = c = \begin{cases} \frac{1}{n} & n-1, \bar{n} \end{cases}
\]

**Proof:** First, by Claim 8, sd-SP implies \( \varphi_{n-1a}(P^{5,1}) = \varphi_{n-1b}(P^{5,1}) = \varphi_{n-1c}(P^{5,1}) = n-1 \) and \( \varphi_{n-1a}(P^{5,1}) + \varphi_{n-1b}(P^{5,1}) + \varphi_{n-1c}(P^{5,1}) = 1 \). Next, suppose \( \varphi_{n-1b}(P^{5,1}) > \frac{1}{2} \). Thus, we have \( \frac{2}{n} > \varphi_{n-1b}(P^{5,1}) + \varphi_{n-1c}(P^{5,1}) > \frac{1}{2} \). Contradiction! Therefore, \( \varphi_{n-1b}(P^{5,1}) = 0 \), and hence \( \varphi_{n-1c}(P^{5,1}) = \frac{2}{n} \). Moreover, ETE implies \( \varphi_{na}(P^{5,1}) = \frac{1}{n} \), \( \varphi_{nb}(P^{5,1}) = 0 \) and \( \varphi_{nc}(P^{5,1}) = \frac{2}{n} \). Last, by ETE and feasibility, we have \( \varphi_{ia}(P^{5,1}) = \frac{1-2x_n}{n-2} = \frac{1}{n} \), \( \varphi_{ib}(P^{5,1}) = \frac{1}{n-2} \) and \( \varphi_{ic}(P^{5,1}) = \frac{1-2x_n}{n-2} = \frac{n-4}{n(n-2)} \) for all \( i = 1, \ldots, n-2 \).

**Claim 13** In profile group V, under both the local elevating and double elevating properties, for each \( m = 2, \ldots, \bar{n}, \bar{n} + 1 \), at \( P^{5,m} = (\bar{P}_1, \ldots, \bar{P}_{m-1}, P_m, \ldots, P_{n-2}, \bar{P}_{n-1}, \bar{P}_n) \), the random assignment \( \varphi(P^{5,m}) \) over \( a \) and \( b \) is specified below

\[
\begin{align*}
a = b = c & \quad \begin{cases} \frac{1}{n} & n-1, \bar{n} \end{cases} \\
1, \ldots, m-1 : & \quad 0 \\
m, \ldots, n-2 & \quad \frac{1}{n(m-1)} \\
n-1, n & \quad \frac{1}{n(m-1)} \\
\gamma(m) & \quad \frac{1-2m-2(n-1)(n-m)}{n(n-1)(n-2)[n-(m-1)(n-m)]}
\end{align*}
\]

where \( \gamma(m) = \frac{2^{m+1}(m+1)n^a+2(m+1)^2n^a-2(m^2+m+1)n^a+1}{n(n-1)(n-2)[n-(m-1)(n-m)]} \).
Proof: The proof of this claim consists of three steps. In the first step, we specify the random assignment over \( a \) and \( b \) for agents \( n-1 \) and \( n \). In the second step, we show that equation \( \gamma(m) \) is decreasing from \( m = 2 \) to \( m = \bar{n} + 1 \), while the third step specifies the random assignment over \( a \) and \( b \) for agents \( 1, \ldots, m-1 \) and \( m, \ldots, n-2 \).

**Step 1:** Given \( 2 \leq m \leq \bar{n} + 1 \), by Claim 9, sd-\( SP \) implies \( \varphi_{n-1,a}(P^{5,m}) = \varphi_{n-1,a}(P^{3,m}) = \frac{1}{n-(m-1)} \) and \( \varphi_{n-1,c}(P^{5,m}) + \varphi_{n-1,b}(P^{5,m}) = \varphi_{n-1,b}(P^{3,m}) + \varphi_{n-1,c}(P^{3,m}) \). By Claim 11, we know \( \varphi_{n-1,c}(P^{5,m}) + \varphi_{n-1,b}(P^{5,m}) < \frac{2}{n-1} \). Suppose \( \varphi_{n-1,b}(P^{5,m}) > 0 \). Since every agent other than \( n-1 \) and \( n \) prefers \( b \) to \( c \), sd-Eff implies \( \varphi_{ia}(P^{5,m}) = 0 \) for all \( i = 1, \ldots, n-2 \). Then, ETE and feasibility imply \( \varphi_{n-1,c}(P^{5,m}) = \frac{1}{2} \). Consequently, we have \( \frac{2}{n-1} > \varphi_{n-1,c}(P^{5,m}) + \varphi_{n-1,b}(P^{5,m}) > \frac{1}{2} \). Contradiction! Therefore, \( \varphi_{n-1,b}(P^{5,m}) = 0 \). Then, by ETE, we have \( \varphi_{na}(P^{5,m}) = \frac{1}{n-(m-1)} \) and \( \varphi_{nb}(P^{5,m}) = 0 \). In conclusion, for each \( m = 2, \ldots, \bar{n}, \bar{n} + 1 \), the random assignment \( \varphi(P^{5,m}) \) over \( a \) and \( b \) for agents \( n-1 \) and \( n \) is specified below:

\[
\begin{array}{c c}
a & b \\
n-1, n : & \frac{1}{n-(m-1)} & 0 \\
\end{array}
\]

**Step 2:** Given \( 3 \leq m \leq \bar{n} + 1 \), we have

\[
\gamma(m) - \gamma(m - 1) = \frac{2n^4 - 2(2m + 1)n^3 + 2(m + 1)^2n^2 - 2(m^2 + m + 1)n + 4}{n(n - 1)(n - 2)[n - (m - 1)](n - m)} - \frac{2n^4 - 2(2m - 1)n^3 + 2m^2n^2 - 2(m^2 - m + 1)n + 4}{n(n - 1)(n - 2)[n - (m - 2)](n - m)} - \frac{2n^2 - mn + 2}{n(n - 1)[n - (m - 2)][n - (m - 1)](n - m)} < 0.
\]

Therefore, \( \gamma(m) \) is decreasing from \( m = 2 \) to \( m = \bar{n} + 1 \).

**Step 3:** For each \( m = 2, \ldots, \bar{n}, \bar{n} + 1 \), we specify the random assignment over \( a \) and \( b \) for agents \( 1, \ldots, m-1 \) and \( m, \ldots, n-2 \).

First, by Claim 12, sd-\( SP \) implies \( \varphi_{ib}(P^{5,2}) + \varphi_{ia}(P^{5,2}) = \varphi_{ib}(P^{5,1}) + \varphi_{ia}(P^{5,1}) = \frac{1}{n} + \frac{1}{n-2} = \gamma(2) \). Next, since sd-Eff implies \( \varphi_{ia}(P^{5,2}) = 0 \), we have \( \varphi_{ib}(P^{5,2}) = \gamma(2) \). Then, by ETE and feasibility, we have \( \varphi_{ia}(P^{5,2}) = \frac{1 - 2^k - \frac{1}{n-2} + \frac{1}{n-3}}{n-3} = \frac{1}{n-2} \) and \( \varphi_{ib}(P^{5,2}) = \frac{1 - (2^k - 1) \gamma(2)}{n-2} \) for all \( i = 2, \ldots, n-2 \).

Next, we adopt an induction hypothesis: Given \( 2 < m \leq \bar{n} + 1 \), for all \( 2 \leq l < m \), the random assignments \( \varphi(P^{5,l}) \) over \( a \) and \( b \) for all agents \( 1, \ldots, l-1 \) and \( l, \ldots, n-2 \) are specified as follows:

\[
\begin{array}{c c}
a & b \\
1, \ldots, l-1 : & 0 & \gamma(l) \\
l, \ldots, n-2 : & \frac{1}{n-(l-1)} & \frac{1-(l-1)\gamma(l)}{n-(l+1)} \\
\end{array}
\]

We specify the random assignment \( \varphi(P^{5,m}) \) over \( a \) and \( b \) for all agents \( 1, \ldots, m-1 \) and \( m, \ldots, n-2 \) to complete the verification of the induction hypothesis. First, by the induction hypothesis, sd-\( SP \)
implies
\[
\varphi_{m-1b}(P^{5,m}) + \varphi_{m-1a}(P^{5,m}) = \varphi_{m-1a}(P^{5,m-1}) + \varphi_{m-1b}(P^{5,m-1}) \\
= \frac{1}{n - (m - 2)} + \frac{1 - (m - 2) \times \gamma(m - 1)}{n - m} \\
= \frac{1}{n - (m - 2)} + \frac{1 - (m - 2) \times \left[2n^4 - 2(2m - 1)n^3 + 2m^2 n^2 - 2(m^2 - m + 1)n + 4\right]}{n(n - 1)(n - 2)(n - (m - 1))} \\
= \frac{2n^4 - 2(2m + 1)n^3 + 2(m + 1)^2n^2 - 2(m^2 + m + 1)n + 4}{n(n - 1)(n - 2)(n - (m - 1))(n - m)} \\
= \gamma(m).
\]

Furthermore, \( ET\) implies \( \varphi_{ib}(P^{5,m}) + \varphi_{ia}(P^{5,m}) = \gamma(m) \) for all \( i = 1, \ldots, m - 1 \).

Next, suppose that there exists \( i^* \in \{1, \ldots, m - 1\} \) such that \( \varphi_{ia}(P^{5,m}) > 0 \). Since every agent other than \( 1, \ldots, m - 1 \) prefers \( a \) to \( b \), \( sd\)-\( Ef \) implies \( \varphi_{ib}(P^{5,m}) = 0 \) for all \( j = m, \ldots, n \). Consequently, \( ET\) and feasibility imply \( \varphi_{ib}(P^{5,m}) = \frac{1}{m - 1} \). Hence, \( \gamma(m) = \varphi_{ib}(P^{5,m}) + \varphi_{ia}(P^{5,m}) > \frac{1}{m - 1} \). Since \( m > 2 \), by Step 2, we have
\[
\gamma(m) \leq \gamma(3) = \frac{2n^4 - 14n^3 + 32n^2 - 26n + 4}{n(n - 1)(n - 2)(n - 3)} < \frac{1}{m - 1}.
\]

Contradiction! Therefore, \( \varphi_{ia}(P^{5,m}) = 0 \) for all \( i = 1, \ldots, m - 1 \). Hence, \( \varphi_{ib}(P^{5,m}) = \gamma(m) \) for all \( i = 1, \ldots, m - 1 \).

Last, by \( ET\) and feasibility, we have \( \varphi_{ia}(P^{5,m}) = \frac{1 - 2m + 1}{n - (m - 1)} = \frac{1}{n - (m - 1)} \) and \( \varphi_{ib}(P^{5,m}) = \frac{1 - (m - 1) \gamma(m)}{n - (m - 1)} \) for all \( i = m, \ldots, n - 2 \). In conclusion, the random assignment \( \varphi(P^{5,m}) \) over \( a \) and \( b \) for all agents \( 1, \ldots, m - 1 \) and \( m, \ldots, n - 2 \) is specified below.

\[
\begin{array}{ccc}
& a & b \\
1, \ldots, m - 1: & 0 & \gamma(m) \\
m, \ldots, n - 2: & \frac{1 - (m - 1) \times \gamma(m)}{n - (m - 1)} & \frac{1}{n - (m - 1)}
\end{array}
\]

This completes the verification of the induction hypothesis, and hence proves the claim. \( \square \)

Claim 14 Under both the local elevating and double elevating properties, for each \( m = 2, \ldots, \bar{n} \), at \( P^{6,m} = (\bar{P}_1, \ldots, \bar{P}_{n-m}, \bar{P}_{n-m+1}, \ldots, \bar{P}_{n-2}, \bar{P}_{n-1}, \bar{P}_n) \), we have \( \varphi_{ia}(P^{6,m}) + \varphi_{ib}(P^{6,m}) = \frac{2}{n} \) for all \( i \in I \).

Proof: We first consider profile \( P^{6,2} = (\bar{P}_1, \ldots, \bar{P}_{n-2}, \bar{P}_{n-1}, \bar{P}_n) \), and show \( \varphi_{ia}(P^{6,2}) + \varphi_{ib}(P^{6,2}) = \frac{2}{n} \) for all \( i \in I \). By Claim 10, \( sd\)-\( SP \) implies (i) \( \varphi_{n-1a}(P^{6,2}) = \varphi_{n-1a}(P^{4,2}) = \frac{2}{n} \), (ii) \( \varphi_{n-1c}(P^{6,2}) + \varphi_{n-1b}(P^{6,2}) = \frac{1}{n} \) under the local elevating property, and (iii) \( \varphi_{n-1c}(P^{6,2}) + \varphi_{n-1b}(P^{6,2}) = \varphi_{n-1b}(P^{4,2}) + \varphi_{n-1c}(P^{4,2}) = \frac{2}{n} \) under the double elevating property. Next, suppose \( \varphi_{n-1b}(P^{6,2}) > 0 \). Since every agent other than \( n - 1 \) and \( n \) prefers \( b \) to \( c \), \( sd\)-\( Ef \) implies \( \varphi_{ic}(P^{6,2}) = 0 \) for all \( i = 1, \ldots, n - 2 \). Then, \( ET\) and feasibility imply \( \varphi_{n-1c}(P^{6,2}) = \frac{1}{2} \). Consequently, we have \( \frac{1}{n} = \varphi_{n-1c}(P^{6,2}) + \varphi_{n-1b}(P^{6,2}) \) under the local elevating property, and \( \frac{2}{n} = \varphi_{n-1c}(P^{6,2}) + \varphi_{n-1b}(P^{6,2}) > \frac{1}{2} \) under the double elevating property. Contradiction! Therefore, \( \varphi_{n-1b}(P^{6,2}) = 0 \). Thus, \( \varphi_{n-1a}(P^{6,2}) + \varphi_{n-1b}(P^{6,2}) = \frac{2}{n} \) and \( \varphi_{ia}(P^{6,2}) + \varphi_{ib}(P^{6,2}) = \frac{2}{n} \).
by $ETE$. Last, by $ETE$ and feasibility, we have $\varphi_{ia}(P^{6,2}) + \varphi_{ib}(P^{6,2}) = \frac{2 - 2 \times \frac{2}{n - 2}}{n - 2} = \frac{2}{n}$ for all $i = 1, \ldots, n - 2$. In conclusion, $\varphi_{ia}(P^{6,2}) + \varphi_{ib}(P^{6,2}) = \frac{2}{n}$ for all $i \in I$.

Next, we adopt an induction hypothesis: Given $2 < m \leq \tilde{n}$, for all $2 \leq l < m$, $\varphi_{ia}(P^{6,l}) + \varphi_{ib}(P^{6,l}) = \frac{2}{n}$ for all $i \in I$. We show $\varphi_{ia}(P^{6,m}) + \varphi_{ib}(P^{6,m}) = \frac{2}{n}$ for all $i \in I$.

By Claim 10, $sd-SP$ implies (i) $\varphi_{n-1a}(P^{6,m}) = \varphi_{n-1a}(P^{4,m}) = \frac{2}{n}$, (ii) $\varphi_{n-1c}(P^{6,m}) + \varphi_{n-1d}(P^{6,m}) = \varphi_{n-1d}(P^{4,m}) + \varphi_{n-1c}(P^{4,m}) = \frac{2}{n}$ under the local elevating property, and (iii) $\varphi_{n-1c}(P^{6,m}) + \varphi_{n-1d}(P^{6,m}) = \varphi_{n-1d}(P^{4,m}) + \varphi_{n-1c}(P^{4,m}) = \frac{2}{n}$ under the double elevating property. Next, suppose $\varphi_{n-1b}(P^{6,m}) > 0$. Since every agent other than $n - 1$ and $n$ prefers $b$ to $c$, $sd-Eff$ implies $\varphi_{ic}(P^{6,m}) = 0$ for all $i = 1, \ldots, n - 2$. Then, $ETE$ and feasibility imply $\varphi_{n-1c}(P^{6,m}) = \frac{1}{2}$. Consequently, we have $\frac{2}{n} = \varphi_{n-1c}(P^{6,m}) + \varphi_{n-1b}(P^{6,m}) > \frac{1}{2}$ under the local elevating property, and $\frac{2}{n} = \varphi_{n-1c}(P^{6,m}) + \varphi_{n-1b}(P^{6,m}) > \frac{1}{2}$ under the double elevating property. Contradiction! Therefore, $\varphi_{n-1b}(P^{6,m}) = 0$. Thus, $\varphi_{n-1a}(P^{6,m}) + \varphi_{n-1b}(P^{6,m}) = \frac{2}{n}$ and $\varphi_{ia}(P^{6,m}) + \varphi_{ib}(P^{6,m}) = \frac{2}{n}$ by $ETE$.

Next, by $sd-SP$ and the induction hypothesis, we have $\varphi_{n-m+1a}(P^{6,m}) + \varphi_{n-m+1b}(P^{6,m}) = \varphi_{n-m+1a}(P^{6,m-1}) + \varphi_{n-m+1b}(P^{6,m-1}) = \frac{2}{n}$. Then, $ETE$ implies $\varphi_{ja}(P^{6,m}) + \varphi_{jb}(P^{6,m}) = \frac{2}{n}$ for all $j = n - m + 1, \ldots, n - 2$. Last, by $ETE$ and feasibility, we have $\varphi_{ia}(P^{6,m}) + \varphi_{ib}(P^{6,m}) = \frac{2 - 2 \times \frac{2}{n - 2}}{n} = \frac{2}{n}$ for all $i = 1, \ldots, n - m$. This completes the verification of the induction hypothesis, and hence proves the claim.

Now, we induce the contradiction for the case of an odd number of agents. Let $n \geq 5$ be an odd integer. Thus, $\tilde{n} = \frac{n - 1}{2}$. Notice that $P^{5,\tilde{n}+1}$ and $P^{6,\tilde{n}}$ differ exactly in preferences of agent $\tilde{n} + 1$, i.e., $P^{5,\tilde{n}+1}_{\tilde{n}+1} = P_i$ and $P^{6,\tilde{n}}_{\tilde{n}+1} = \tilde{P}_i$ in Definition 3 or 4. Then, $sd-SP$ implies $\varphi_{\tilde{n}+1a}(P^{5,\tilde{n}+1}) + \varphi_{\tilde{n}+1b}(P^{5,\tilde{n}+1}) = \varphi_{\tilde{n}+1a}(P^{6,\tilde{n}}) + \varphi_{\tilde{n}+1b}(P^{6,\tilde{n}})$. Thus, by Claims 13 and 14, we have

$$0 = \left[ \varphi_{\tilde{n}+1a}(P^{5,\tilde{n}+1}) + \varphi_{\tilde{n}+1b}(P^{5,\tilde{n}+1}) \right] - \left[ \varphi_{\tilde{n}+1a}(P^{6,\tilde{n}}) + \varphi_{\tilde{n}+1b}(P^{6,\tilde{n}}) \right]$$

$$= \left[ \frac{1}{n - \tilde{n}} + \frac{1 - \tilde{n} \times \gamma(\tilde{n} + 1)}{n - \tilde{n} + 2} \right] - \frac{2}{n} = \frac{-4}{(n - 3)(n - 2)(n - 1)n}. \text{ Contradiction!}$$

Therefore, in the case of an odd number of agents, domain $\mathbb{D}$ satisfying the local elevating property or the double elevating property admits no $sd-SP$, $sd-Eff$ and $ETE$ rule. This completes the proof of Lemma 1

## C Proof of Lemma 2

Let $\mathbb{D}$ be a weakly connected domain violating both the local elevating and double elevating properties.

**Claim 15** Given distinct $P_i, P_i', P_i'' \in \mathbb{D}$, if $P_i \approx P_i'$ and $P_i' \approx P_i''$, then $P_i \approx P_i''$.

To prove Claim 15, it suffices to show the following two symmetric statements:

(i) Given $a, b \in A$, if $r_k(P_i) = r_{k+1}(P_i') = b$ and $r_{k+1}(P_i) = r_k(P_i') = a$ for some $1 \leq k < n$, then we have either $r_k(P_i'') = a$, $r_{k+1}(P_i'') = b$ and $B(P_i, b) = B(P_i', a) = B(P_i'', a)$, or $r_k(P_i'') = b$, $r_{k+1}(P_i'') = a$ and $B(P_i, b) = B(P_i', a) = B(P_i'', b)$.
(ii) Given $a, b \in A$, if $r_k(P'_i) = r_{k+1}(P''_i) = b$ and $r_{k+1}(P'_i) = r_k(P''_i) = a$ for some $1 \leq k < n$, then we have either $r_k(P_i) = a$, $r_{k+1}(P_i) = b$ and $B(P'_i, b) = B(P''_i, a) = B(P_i, a)$, or $r_k(P_i) = b$, $r_{k+1}(P_i) = a$ and $B(P'_i, b) = B(P''_i, a) = B(P_i, b)$.

Since both statements are symmetric, we focus on showing statement (i). Since $P_i \approx P'_i$, we can identify several pair(s) of objects $(b_l, a_l) : l = 1, \ldots, t$ that are locally switched across $P_i$ and $P'_i$, i.e., there exist $1 \leq k_1 < k_2 < \cdots < k_t < n$ such that $b_l = r_{k_l}(P_i) = r_{k_l+1}(P'_i)$, $a_l = r_{k_{l+1}}(P_i) = r_{k_{l+1}+1}(P'_i)$, $l = 1, \ldots, t$, and $[x, y] \iff [x, y]$ for all $(x, y) \notin \{b_l, a_l : l = 1, \ldots, t\}$. We first show that statement (i) holds for the pair $(b_1, a_1)$.

Since $a_1 = r_{k_1}(P'_i)$ and $b_1 = r_{k_1+1}(P'_i)$, $P'_i \approx P''_i$ implies $a_1 \in \{r_{k_1-1}(P''_i), r_{k_1}(P''_i), r_{k_1+1}(P''_i)\}$ and $b_1 \in \{r_{k_1}(P''_i), r_{k_1+1}(P''_i), r_{k_2+2}(P''_i)\}$. First, suppose $a_1 \notin \{r_{k_1}(P''_i), r_{k_1+1}(P''_i)\}$. Thus, $a_1 = r_{k_1-1}(P''_i)$. Then, by $P'_i \approx P''_i$, we know (i) $r_{k_1-1}(P''_i) = x$, (ii) $x \neq a_1$ and $a_1 \neq x$, and (iii) $x \neq b_1$. Note that $(b_1, a_1)$ is the first pair (the highest ranked pair) which is locally switched across $P_i$ and $P'_i$. Then, $P_i \approx P'_i$ implies $r_{k_1-1}(P_i) = r_{k_1-1}(P'_i) = x$. We next assert $B(P_i, x) = B(P'_i, x) = B(P''_i, a_1)$. The first equality holds evidently. Suppose $B(P'_i, x) \neq B(P''_i, a_1)$. Since $|B(P'_i, x)| = |B(P''_i, a_1)| = k_1 - 2$, $B(P'_i, x) \neq B(P''_i, a_1)$ implies that there exists $y \in B(P''_i, a_1) \setminus B(P'_i, x)$. Thus, $y \neq a_1$, $y \neq x$, and $x \neq y$. Since $a_1 \neq y$, we have $y \neq P'_i x$. Then, $P'_i \approx P''_i$ implies $x \neq P'_i x$. Therefore, $B(P_i, x) = B(P'_i, x) = B(P''_i, a_1)$. Furthermore, since $b_1 \in \{r_{k_1}(P''_i), r_{k_1+1}(P''_i), r_{k_1+2}(P''_i)\}$ and $r_{k_1}(P''_i) = x \neq b_1$, we have two cases: (1) $r_{k_1+1}(P''_i) = b_1$ and (2) $r_{k_1+2}(P''_i) = b_1$. In case (1), we have an instance of the local elevating property specified in Table 4 below.

\[
\begin{array}{cccccc}
& k_1 & k_1 & k_1 & k_1 & k_1 \\
\text{P}_i: & \cdots & \cdots & \cdots & \cdots & \cdots \\
& x & b_1 & a_1 & \cdots & \cdots \\
& B(P_i, x) = B(P'_i, x) & B(P'_i, x) = B(P''_i, a_1) \\
\text{P}'_i: & \cdots & \cdots & \cdots & \cdots & \cdots \\
& x & a_1 & b_1 & \cdots & \cdots \\
& B(P'_i, x) = B(P''_i, a_1) \\
\text{P}''_i: & \cdots & \cdots & \cdots & \cdots & \cdots \\
& a_1 & x & b_1 & \cdots & \cdots \\
\end{array}
\]

Table 4: An instance of the local elevating property

In case (2), by $P'_i \approx P''_i$, we know $r_{k_1+2}(P''_i) = r_{k_1+1}(P'_i) = b_1$ and $r_{k_1+1}(P''_i) = r_{k_1+2}(P'_i) = z$. Consequently, we have an instance of the double elevating property specified in Table 5 below.

\[
\begin{array}{cccccc}
& k_1 & k_1 & k_1 & k_1 & k_1 \\
\text{P}_i: & \cdots & \cdots & \cdots & \cdots & \cdots \\
& x & b_1 & a_1 & \cdots & \cdots \\
& B(P_i, x) = B(P'_i, x) & B(P'_i, x) = B(P''_i, a_1) \\
\text{P}'_i: & \cdots & \cdots & \cdots & \cdots & \cdots \\
& x & a_1 & b_1 & \cdots & \cdots \\
& B(P'_i, x) = B(P''_i, a_1) \\
\text{P}''_i: & \cdots & \cdots & \cdots & \cdots & \cdots \\
& a_1 & x & z & b_1 & \cdots \\
\end{array}
\]

Table 5: An instance of the double elevating property

Hence, in each case, we induce a contradiction. Therefore, $a_1 \in \{r_{k_1}(P''_i), r_{k_1+1}(P''_i)\}$.

Next, suppose $b_1 \notin \{r_{k_1}(P''_i), r_{k_1+1}(P''_i)\}$. Thus, $b_1 = r_{k_1+2}(P''_i)$. Then, by $P'_i \approx P''_i$, we know $r_{k_1+1}(P''_i) = r_{k_1+2}(P'_i) = x$, and hence $x \neq a_1$. Furthermore, since $a_1 \in \{r_{k_1}(P''_i), r_{k_1+1}(P''_i)\}$, we have $r_{k_1}(P''_i) = a_1$. Thus, $a_1 = r_{k_1}(P'_i) = r_{k_1}(P''_i)$. We next assert $B(P_i, b_1) = B(P'_i, a_1) = \cdots$.
In case (2), since \( b \), we have an instance of the local elevating property specified in Table 6 below.

<table>
<thead>
<tr>
<th>( k_1 )</th>
<th>( k_1 + 1 )</th>
<th>( k_1 + 2 )</th>
<th>( k_1 + 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>( x )</td>
<td>( b_1 )</td>
<td>( \cdots )</td>
</tr>
</tbody>
</table>

Table 6: An instance of the local elevating property

In case (2), since \( P_i \approx P_i' \), we have \( r_{k_1+2}(P_i') = r_{k_1+2}(P_i) = x \) and \( r_{k_1+3}(P_i') = r_{k_1+2}(P_i) = x \). Consequently, we have an instance of the double elevating property specified in Table 7 below.

<table>
<thead>
<tr>
<th>( k_1 )</th>
<th>( k_1 + 1 )</th>
<th>( k_1 + 2 )</th>
<th>( k_1 + 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>( x )</td>
<td>( b_1 )</td>
<td>( \cdots )</td>
</tr>
</tbody>
</table>

Table 7: An instance of the double elevating property

Hence, in each case, we induce a contradiction. Therefore, \( b_1 \in \{ r_{k_1}(P_i'''), r_{k_1+1}(P_i'') \} \). In conclusion, we have either \( r_{k_1}(P_i''') = a_1 \) and \( r_{k_1+1}(P_i''') = b_1 \), or \( r_{k_1}(P_i''') = b_1 \) and \( r_{k_1+1}(P_i''') = a_1 \).

Last, we show \( [r_{k_1}(P_i''') = a_1] \Rightarrow [B(P_i, b_1) = B(P_i', a_1) = B(P_i'', a_1)], \) and \( [r_{k_1}(P_i''') = b_1] \Rightarrow [B(P_i, b_1) = B(P_i', a_1) = B(P_i'', b_1)]. \) It is evident that \( B(P_i, b_1) = B(P_i', a_1) \).

We first assume \( r_{k_1}(P_i''') = a_1 \). Suppose \( B(P_i', a_1) \neq B(P_i'', a_1) \). Since \( |B(P_i', a_1)| = |B(P_i'', a_1)| = k_1 - 1 \), \( B(P_i', a_1) \neq B(P_i'', a_1) \) implies that there exists \( y \in B(P_i'', a_1) \backslash B(P_i', a_1) \). Consequently, we have \( y \neq a_1 \) and \( P_i''' \) \( y \) which by \( P_i''' \approx P_i'' \) imply \( a_1 \) \( y \) and \( y \). This contradicts the fact \( r_{k_1}(P_i') = r_{k_1}(P_i'') = a_1 \). Therefore, \( B(P_i, b_1) = B(P_i', a_1) = B(P_i'', a_1) \).

We next assume \( r_{k_1}(P_i'') = b_1 \). Thus, \( a_1 \) \( P_i''' \) \( b_1 \) and \( a_1 \) \( P_i''' \) \( a_1 \). Suppose \( B(P_i', a_1) \neq B(P_i'', b_1) \). Since \( |B(P_i', a_1)| = |B(P_i'', b_1)| = k_1 - 1 \), \( B(P_i', a_1) \neq B(P_i'', b_1) \) implies that there exists \( y \in B(P_i'', b_1) \backslash B(P_i', a_1) \). Consequently, we have \( y \neq b_1 \), \( P_i''' \) \( b_1 \) and \( a_1 \) \( P_i''' \) \( y \). Furthermore, since \( b_1 \) \( P_i'' \) \( a_1 \), we have \( y \) \( P_i'' \) \( a_1 \). Consequently, \( P_i''' \approx P_i'' \) implies \( a_1 \) \( y \) and \( y \). This contradicts the fact \( a_1 \) \( P_i'' \) \( b_1 \) and \( a_1 \) \( P_i''' \) \( a_1 \). Therefore, \( B(P_i, b_1) = B(P_i', a_1) = B(P_i'', b_1) \). This completes the verification of statement (i) for the pair \( (b_1, a_1) \).

Next, we provide an induction hypothesis: Given \( 1 < m \leq t \), statement (i) holds for all pairs \( (b_1, a_1), \ldots, (b_{m-1}, a_{m-1}) \). We show that statement (i) holds for \( (b_m, a_m) \).

Since \( a_m = r_{k_m}(P_i') \) and \( b_m = r_{k_m+1}(P_i') \), \( P_i' \approx P_i''' \) implies \( a_1 \in \{ r_{k_m-1}(P_i''), r_{k_m}(P_i''), r_{k_m+1}(P_i'') \} \) and \( b_1 \in \{ r_{k_m}(P_i''), r_{k_m+1}(P_i''), r_{k_m+2}(P_i'') \} \). Suppose \( a_m \notin \{ r_{k_m}(P_i''), r_{k_m+1}(P_i'') \} \). Thus, \( a_m = r_{k_m-1}(P_i'') \). Then, by \( P_i' \approx P_i''' \), we know (i) \( r_{k_m}(P_i'') = r_{k_m-1}(P_i') \), (ii) \( x \) \( P_i''' \), \( a_m \), and
we know \( b_1 \in \{ r_{km}(P'_i), r_{km+1}(P'_i), r_{km+2}(P'_i) \} \) and \( x = r_{km}(P'_i) \), we know \( b_1 \in \{ r_{km+1}(P'_i), r_{km+2}(P'_i) \} \). We next assert \( r_{km-1}(P_i) = x \). Suppose not, i.e., \( r_{km-1}(P_i) \equiv y \neq x \). Since \( r_{km}(P_i) = b_m \neq x \), \( P_i \approx P'_i \) implies \( r_{km-2}(P_i) = r_{km-1}(P_i) = x \) and \( r_{km-1}(P_i) = r_{km-2}(P'_i) = y \). Then, it must be the case \((b_{m-1}, a_{m-1}) = (x, y)\), and the induction hypothesis implies \( x \in \{ r_{km-2}(P''_i), r_{km-1}(P''_i) \} \) which contradicts \( r_{km}(P''_i) = x \). Therefore, \( r_{km-1}(P_i) = x \). Then, symmetric to the argument related to \( a_1 \) above, we have \( B(P_1, x) = B(P'_1, x) = B(P'_1, a_m) \), and induce an instance of the local elevating property if \( r_{km+1}(P''_i) = b_m \), and an instance of the double elevating property if \( r_{km+2}(P''_i) = b_m \), which both contradict the hypothesis of Lemma 2. Hence, \( a_m \in \{ r_{km}(P'_i), r_{km+1}(P'_i) \} \).

Next, suppose \( b_m \not\in \{ r_{km}(P''_i), r_{km+1}(P''_i) \} \). Thus, \( b_m = r_{km+2}(P''_i) \). Then, by \( P'_i \approx P''_i \), we know \( r_{km+1}(P'_i) = r_{km+2}(P'_i) \equiv x \), and hence \( x \neq a_m \). Since \( a_m \in \{ r_{km}(P'_i), r_{km+1}(P'_i) \} \) and \( x = r_{km}(P''_i) \), we have \( r_{km}(P''_i) = a_m \). Moreover, since \( P_i \approx P'_i \) and \( r_{km+1}(P'_i) = a_m \), we know \( x \in \{ r_{km+2}(P_i), r_{km+3}(P_i) \} \). Then, symmetric to the argument related to \( b_1 \) above, we have \( B(P_i, b_m) = B(P'_i, a_m) = B(P''_i, a_m) \), and induce an instance of the local elevating property if \( r_{km+2}(P_i) = x \), and an instance of the double elevating property if \( r_{km+3}(P_i) = x \), which both contradict the hypothesis of Lemma 2. Therefore, \( b_m \in \{ r_{km}(P''_i), r_{km+1}(P''_i) \} \). In conclusion, we have either either \( r_{km}(P''_i) = a_m \) and \( r_{km+1}(P''_i) = b_m \), or \( r_{km}(P'') = b_m \) and \( r_{km+1}(P''_i) = a_m \).

Last, symmetric to the argument related to \((b_1, a_1)\) above, we assert \([r_{km}(P''_i) = a_m] \Rightarrow [B(P_i, b_m) = B(P'_i, a_m) = B(P''_i, a_m)] \) and \([r_{km}(P''_i) = b_m] \Rightarrow [B(P_i, b_m) = B(P'_i, a_m) = B(P''_i, b_m)] \). This completes the verification of the induction hypothesis. Hence, we prove statement (i) and Claim 15.

Last, since domain \( D \) is weakly connected, we know that every pair of distinct preferences is connected via a path. Then, Claim 15 immediately implies that all preferences of \( D \) are pairwise neighbors. Consequently, for an arbitrary pair of distinct preferences \( P_i, P'_i \in D \), if two objects are oppositely ranked across \( P_i \) and \( P'_i \), say \( a P_i b \) and \( b P'_i a \), they must be consecutively ranked in both \( P_i \) and \( P'_i \), i.e., \( a = r_k(P_i) = r_{k+1}(P'_i) \) and \( b = r_{k+1}(P_i) = r_k(P'_i) \) for some \( 1 \leq k < n \). Furthermore, by statements (i) and (ii) in the proof of Claim 15, we know that \( a \) and \( b \) also occupy the \( k \)-th and \((k + 1)\)-th ranking positions in every other preference of \( D \). Therefore, \( D \) must be a restricted tier domain. This completes the proof of Lemma 2.

## D Details related to Remark 6

Let domain \( D \) satisfy the local elevating property (respectively, the double elevating property). We show that \( D \) admits no \( sd-SP \), \( sd-Eff \) and \( sd-EF \) rule.

Assume w.l.o.g. that \( D \) includes preferences \( \hat{P}_i, P_i \) and \( \hat{P}_i \) of Table 1 (respectively, Table 2). Suppose that there exists an \( sd-SP \), \( sd-Eff \) and \( sd-EF \) rule \( \varphi : D^n \rightarrow L \). We construct two preference profiles which only consist of preferences \( \hat{P}_i, P_i \) and \( \hat{P}_i \) of Table 1 (respectively, Table 2):

- \( P^1 = (P_1, P_2, \ldots, P_{n-1}, \hat{P}_n) \): Agent \( n \) reports \( \hat{P}_i \), while everyone else reports \( P_i \).
- \( P^2 = (\hat{P}_1, P_2, \ldots, P_{n-1}, \hat{P}_n) \): Agent \( n \) reports \( \hat{P}_i \), agent 1 reports \( \hat{P}_1 \), while everyone else reports \( P_i \).

**Step 1:** For notational convenience, let \( B \equiv B(\hat{P}_1, a) = B(P_i, a) = B(\hat{P}_i, b) \). Evidently, \( sd-EF \) and feasibility imply \( \varphi_{1a}(P^1) = \frac{k-1}{n} \) and \( \varphi_{1b}(P^2) = \frac{k+1}{n} \) for all \( i \in I \). Then, by \( sd-SP \), we have \( \varphi_{1a}(P^1) + \varphi_{1b}(P^1) = \varphi_{1b}(P^2) + \varphi_{1a}(P^2) \).
Step 2: We specify the random assignments \( \varphi(P^1) \) and \( \varphi(P^2) \) over \( a \) and \( b \) by \( sd-Eff \) and \( sd-EF \) in the following two claims.

Claim 16 The random assignment \( \varphi(P^1) \) over \( a \) and \( b \) is specified below.

\[
\begin{array}{cc}
1 : & 0 \\
\cdots & \frac{1}{n} \\
\cdots & \frac{1}{n-1} \\
n : & 0 \\
\end{array}
\]

First, \( sd-EF \) and feasibility imply \( \varphi_{ia}(P^1) = \frac{1}{n} \) for all \( i \in I \). Next, \( sd-Eff \) implies \( \varphi_{nb}(P^1) = 0 \). Then, by \( sd-EF \) and feasibility, we have \( \varphi_{ib}(P^1) = \frac{1}{n-1} \) for all \( i = 1, \ldots, n - 1 \). This completes the verification of the claim.

Claim 17 The random assignment \( \varphi(P^2) \) over \( a \) and \( b \) is specified below.

\[
\begin{array}{cc}
1 : & 0 \\
\cdots & \frac{2n-3}{(n-1)^2} \\
\cdots & \frac{n-2}{(n-1)^2} \\
n : & 0 \\
\end{array}
\]

First, \( sd-Eff \) implies \( \varphi_{ia}(P^2) = 0 \). Then, by \( sd-EF \) and feasibility, we have \( \varphi_{ia}(P^2) = \frac{1}{n-1} \) for all \( i = 2, \ldots, n \). Second, \( sd-Eff \) implies \( \varphi_{nb}(P^2) = 0 \). Then, feasibility implies \( \varphi_{ib}(P^2) + \sum_{i=2}^{n-1} \varphi_{ib}(P^2) = 1 \). Last, since \( sd-EF \) implies \( \varphi_{ia}(P^2) + \varphi_{ib}(P^2) = \varphi_{ia}(P^2) + \varphi_{ib}(P^2) \) for all \( i = 2, \ldots, n - 1 \), and \( \varphi_{ib}(P^2) = \varphi_{jb}(P^2) \) for all \( i, j \in \{2, \ldots, n - 1\} \), we calculate \( \varphi_{ib}(P^2) = \frac{n-2}{(n-1)^2} \) for all \( i = 2, \ldots, n - 1 \), and \( \varphi_{ib}(P^2) = \frac{2n-3}{(n-1)^2} \). This completes the verification of the claim.

Last, according to Claims 16 and 17, we have

\[
\varphi_{ia}(P^1) + \varphi_{ib}(P^1) = \frac{1}{n} + \frac{1}{n-1} > \frac{2n-3}{(n-1)^2} = \varphi_{ib}(P^2) + \varphi_{ia}(P^2).
\]

This contradicts Step 1. Therefore, \( \mathbb{D} \) admits no \( sd-SP \), \( sd-Eff \) and \( sd-EF \) rule.

E Proof of Lemma 3

Suppose that \( \mathbb{D}(\Omega_K) \) satisfies the local elevating property (respectively, the double elevating property), including preferences \( \hat{P}_i \), \( P_i \) and \( \bar{P}_i \) of Table 1 (respectively, Table 2). Recall the algorithm. According to objects \( a, b \) and \( c \) in Table 1 (respectively, Table 2), there must exist a unique step \( 1 \leq k \leq K \) such that (i) \( a, b, c \) are included in a block \( A_t \in \mathbb{A}_{k-1} \), and (ii) \( A_t \) breaks into \( A^1_t \) and \( A^2_t \) such that two objects of \{\( a, b, c \)\} and the third one are separated in \( A^1_t \) and \( A^2_t \). We assume w.l.o.g. that two objects of \{\( a, b, c \)\} are included in \( A^1_t \), and the third one is included in \( A^2_t \). Note that \( \mathbb{D}(\Omega_K) \subseteq \mathbb{D}(\Omega_{K-1}) \subseteq \cdots \subseteq \mathbb{D}(\Omega_k) \), and in every preference of \( \mathbb{D}(\Omega_k) \), either all objects of \( A^1_t \) rank above all objects of \( A^2_t \), or all objects of \( A^2_t \) rank above all objects of \( A^1_t \). Consequently, we have \([a, b \in A^1_t] \Rightarrow [\hat{P}_i \notin \mathbb{D}(\Omega_K)], [a, c \in A^1_t] \Rightarrow [P_i \notin \mathbb{D}(\Omega_K)], \) and \([b, c \in A^1_t] \Rightarrow [\bar{P}_i \notin \mathbb{D}(\Omega_K)]\), which contradict the hypothesis of Lemma 3. Therefore, \( \mathbb{D}(\Omega_K) \) avoids both the local elevating and double elevating properties.
F Proof of Proposition 1

We show that \( \mathbb{D}(\Omega_K) \) is equivalent to a *sequentially dichotomous domain* of Liu (2018). Then, by Theorem 1 of Liu (2018), we know that the PS rule is *sd-SP* on \( \mathbb{D}(\Omega_K) \) which hence completes the verification of Proposition 1.

We first introduce two new notions which are used in the definition of a *sequentially dichotomous domain*. First, a sequence of partitions \( (A_k)^T_{k=0} \) is called a *dichotomous path* if it satisfies the following two conditions:

1. \( A_0 = \{ A \} \) and \( A_T = \{ \{ a \} : a \in A \} \).

2. For each \( 1 \leq k \leq T \), some \( A_t \in A_{k-1} \) breaks into nonempty \( A_1^t \) and \( A_2^t \), and \( A_k = \{ A_1^t, A_2^t \} \cup A_{k-1}\{ A_t \} \).

Next, given a partition \( A_k = \{ A_1, \ldots, A_t \} \), a preference \( P_t \) *respects* \( A_k \) if for distinct \( A_p, A_q \in A_k \), either all objects of \( A_p \) rank above all objects of \( A_q \) in \( P_t \), or vice versa, i.e., either \( a P_t b \) for all \( a \in A_p \) and \( b \in A_q \), or \( b P_t a \) for all \( a \in A_p \) and \( b \in A_q \). Now, we introduce the definition of a *sequentially dichotomous domain*. A domain \( \mathbb{D} \) is a sequentially dichotomous domain if there exists a dichotomous path \( (A_k)^T_{k=0} \) such that we have \( P_t \in \mathbb{D} \Leftrightarrow P_t \) respects all \( A_0, \ldots, A_T \).

Now, we start to prove Proposition 1. At each step of the algorithm, we generate a partition. Thus, we have \( A_0 = \{ A \} \) and \( A_1, \ldots, A_K \). At the termination step \( K \), assume w.l.o.g. that \( A_K = \{ A_1, \ldots, A_{t+1}, \ldots, A_{K+1} \} \) where \( |A_k| = 2 \) for all \( k = 1, \ldots, t \), and \( |A_k| = 1 \) for all \( s = t+1, \ldots, K+1 \). Moreover, from \( k = 1 \) to \( k = t \), we continue to consecutively break \( A_k \) into two singleton subsets, and dichotomously refine all restricted tier structures of \( \Omega_K \) accordingly. Thus, for each \( k = 1, \ldots, t \), we have partition \( A_{K+k} \) which replaces \( A_k \in A_{K+k-1} \) by the two corresponding singleton subsets, and the set of tier structures \( \Omega_{K+k} \) which collects the dichotomous refinements of each tier structure in \( \Omega_{K+k-1} \). Thus, \( A_{K+t} = \{ \{ a \} : a \in A \} \), and we have a dichotomous path \( (A_0, A_1, \ldots, A_K, A_{K+1}, \ldots, A_{K+t}) \). Now, to show that \( \mathbb{D}(\Omega_K) \) is equivalent to the corresponding sequentially dichotomous domain, it suffices to show the following two conditions:

(i) Given \( P_t \in \mathbb{D}(\Omega_K) \) and \( 0 \leq k \leq K + t \), \( P_t \) respects \( A_k \).

(ii) Given \( P_t \notin \mathbb{D}(\Omega_K) \), there exists \( 0 \leq k \leq K + t \) such that \( P_t \) does not respect \( A_k \).

Given \( P_t \in \mathbb{D}(\Omega_K) \), we know \( P_t \in \mathbb{D}(\Omega_k) \) for all \( k = 0, 1, \ldots, K \). Next, given \( 0 \leq k \leq K \), we know \( P_t \in \mathbb{D}(P) \) for some \( P \in \mathbb{P} \). Then, Definition 2 implies that \( P_t \) respects \( A_k \). Therefore, \( P_t \) respects \( A_0, A_1, \ldots, A_K \). Furthermore, since blocks \( A_1, \ldots, A_t \) of \( A_K \) consecutively break into two singleton subsets from \( A_K \) to \( A_{K+t} \), it is evident that \( P_t \) continues to respect \( A_{K+k} \) for all \( k = 1, \ldots, t \). This completes the verification of condition (i).

Given \( P_t \notin \mathbb{D}(\Omega_K) \), since \( P_t \in \mathbb{P} \equiv \mathbb{D}(\Omega_0) \), there exists \( 0 < k \leq K \) such that \( P_t \notin \mathbb{D}(\Omega_{k-1}) \) and \( P_t \notin \mathbb{D}(\Omega_k) \). Accordingly, let \( P_t \in \mathbb{D}(P) \) for some \( P \in \mathbb{P} \). Thus, \( P_t \) respects \( A_{k-1} \). Furthermore, assume that block \( A_s \in A_{k-1} \) breaks into \( A_{s}^1 \) and \( A_{s}^2 \), and let \( \overline{P} \) and \( P \) be the corresponding dichotomous refinements of \( P \) in Step \( k \) of the algorithm. Since \( 0 < k \leq K \), it is true that \( |A_s| \geq 3 \), and either \( |A_s^1| \geq 2 \) or \( |A_s^2| \geq 2 \). Since \( P_t \notin \mathbb{D}(\Omega_k) \), it is true that \( P_t \notin \mathbb{D}(\overline{P}) \cup \mathbb{D}(P) \). Then, there exist \( a, b \in A_s^1 \) and \( c \in A_s^2 \), or \( a, b \in A_s^2 \) and \( c \in A_s^1 \) such that \( a P_t c \) and \( c P_t b \). Consequently, \( P_t \) does not respect \( A_k \). This completes the verification of condition (ii), and hence proves Proposition 1.
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